The Manifold of a Planar Array and its Effects on the Accuracy of Direction-Finding Systems

by

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ABSTRACT

The authors are concerned with the accuracy of azimuth and elevation estimates provided by a planar array of sensors and its relation to the array manifold differential geometry. The paper builds on previously published results regarding the influence of manifold differential geometry on the detection and resolution capabilities of linear arrays.

The manifold of a planar array is introduced in terms of two families of constant-azimuth and constant-elevation curves and their differential geometry is analyzed as a function of array configuration. Circular approximation is subsequently employed to derive novel expressions for the Cramer-Rao Lower Bound on azimuth and elevation estimates in terms of the arc lengths and first curvatures of the respective constant-parameter manifold curves. The scenarios considered include the cases of a single emitter as well as that of two closely spaced uncorrelated emitters of arbitrary powers. The results obtained are demonstrated for the case of two practical array configurations.

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1. INTRODUCTION

Consider the popular model for the output $\mathbf{x}(t)$ of an array of $N$ sensors receiving $M$ narrowband signals:

$$
\mathbf{x}(t) = \mathbf{A}(p) \mathbf{m}(t) + \mathbf{n}(t) \quad (1)
$$

In this model, $\mathbf{x}(t) \in \mathbb{C}^{N \times 1}$ is the noisy array output, $\mathbf{m}(t) \in \mathbb{C}^{M \times 1}$ is the vector of complex (narrowband) signal envelopes, $\mathbf{n}(t) \in \mathbb{C}^{N \times 1}$ is additive noise, and the matrix $\mathbf{A}(p) \in \mathbb{C}^{N \times M}$ has the following structure:

$$
\mathbf{A}(p) = [\mathbf{a}(p_1), \ldots, \mathbf{a}(p_M)] \quad (2)
$$

where $p = [p_1, \ldots, p_M]$ are the real source location parameters (such as azimuth and elevation bearings) which are to be estimated. Vector $\mathbf{a}(p_i)$ is the so-called Source Position Vector (SPV) or manifold vector which represents the complex array response to a single source with location parameters $p_i$, and hence is a direct function of the array configuration. The set of all source position vectors forms a locus, or vector continuum, termed the array manifold and defined as $\{\mathbf{a}(p); p \in \Omega\}$ where $\Omega$ is the parameter space.

Note that the parameter space $\Omega$ of the array manifold is the set of all possible source angle-of-arrival parameters, $p$. For example, in an azimuth-only system ($p = \theta$), $\Omega$ is the interval $[0, 2\pi]$ on the real line $\mathbb{R}^1$ and consequently the image of the single-parameter array manifold $\mathbf{a}(\theta)$ traces out a single closed curve winding through $\mathbb{C}^N$ as a function of $\theta$. Similarly in an azimuth-elevation system ($p = (\theta, \phi)$), $\Omega$ is the region $[0, 2\pi] \times [0, \frac{\pi}{2}]$ on the real plane $\mathbb{R}^2$ and so the image of the two-parameter array manifold $\mathbf{a}(\theta, \phi)$ traces out a surface in $\mathbb{C}^N$ as a function of $\theta$ and $\phi$. 

List of Symbols

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<tr>
<td>$A, a$</td>
<td>Scalar</td>
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<tr>
<td>$\mathbf{A}, \mathbf{a}$</td>
<td>Vector</td>
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<tr>
<td>$^T \mathbf{A}$</td>
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<td>$</td>
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<td>$</td>
<td>a</td>
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<tr>
<td>$\mathbb{R}^N$</td>
<td>N-dimensional real space</td>
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<td>$\mathbb{C}^N$</td>
<td>N-dimensional complex space</td>
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<td>$p$</td>
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<td>$s_p(p)$</td>
<td>Rate of change of p-curve length</td>
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<td>$\Delta s$</td>
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<td>$\mathbf{I}$</td>
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<td>$\mathbf{P}$</td>
<td>Projection operator</td>
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<td>Orthogonal-projection operator</td>
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<td>$\text{Re}(\cdot)$</td>
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<tr>
<td>$\text{sum}(a)$</td>
<td>Sum of vector elements</td>
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<tr>
<td>$\mathbb{C}^N[\mathbf{A}]$</td>
<td>Space spanned by columns of $\mathbf{A}$</td>
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<tr>
<td>$\odot$</td>
<td>Elemental (Hadamard) product operator</td>
</tr>
<tr>
<td>$\exp{a \odot \mathbf{A}}$</td>
<td>Element by element exponential</td>
</tr>
<tr>
<td>$\mathbf{a}_k^\text{th}$</td>
<td>Element by element $k^{th}$ power</td>
</tr>
<tr>
<td>$\mathbf{P}_i$</td>
<td>Power of the $i^{th}$ signal</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>Noise power</td>
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<tr>
<td>$\dot{a}$</td>
<td>Derivative $\partial a / \partial p$</td>
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<tr>
<td>$d$</td>
<td>Derivative $\partial a / \partial s$</td>
</tr>
<tr>
<td>$\mathbf{u}_i(p)$</td>
<td>$i^{th}$ coordinate vector of p-curve</td>
</tr>
<tr>
<td>$\kappa_i(p)$</td>
<td>$i^{th}$ curvature of p-curve</td>
</tr>
<tr>
<td>$\mathbf{U}$</td>
<td>Matrix of coordinate vectors $\mathbf{u}_i$</td>
</tr>
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DF: Direction Finding

DOA: Direction of Arrival
1.1. The importance of the array manifold

The significance of the array manifold in the operation of a direction-of-arrival estimation system becomes apparent when it is noted that the procedure of all signal subspace-type superresolution algorithms involves a search of the array manifold in order to locate the true SPVs according to some chosen criterion. For example, in the case of the MuSIC algorithm [1], the manifold is searched for those $a(p)$ which are closest (in a Euclidian sense) to the signal subspace.

The signal subspace is defined as the space spanned by the $M$ Source-Position-Vectors $\{a(p_1) \ldots a(p_M)\}$ and can only be exactly estimated from the array output on the basis of either an infinite number of snapshots (i.e. $L = \infty$), or, an infinite Signal-to-Noise ratio (i.e. $n(t) = 0$). Of course, by definition, under such asymptotic conditions the array manifold actually intersects the signal subspace at points corresponding to the true parameter values $p_i$. This allows complete resolution of the sources for arbitrarily small angular separations — hence the title "superresolution".

In practice, however, due to the availability of a limited number of snapshots (finite observation interval), and the presence of noise, the array manifold does not intersect the estimated signal subspace but merely approaches it at points corresponding to false parameter values, $\hat{p}_i$, thus resulting in loss of resolution and accuracy.

From the above discussion it is apparent that the array manifold — and hence the array configuration — plays an integral role in the operation of signal subspace algorithms and can set fundamental limits on the quality of the array's performance. The influence of the array manifold on system performance may be assessed quantitatively by determining its shape and orientation in relation to the signal subspace, a task which can be accomplished effectively through a study of the manifold's differential geometry.

This was demonstrated by the authors in a recent paper [2] where the detection and resolution capabilities inherent to the manifold of a linear array of isotropic sensors were investigated. It was shown that the thresholds of detection and resolution for a linear array are directly influenced by the rate of change of arc length and the first curvature of the array manifold curve $a(\theta)$ in an azimuth-estimation system.

1.2. Objectives of paper

The objective of this paper is to build upon previous results and to determine, in a quantitative manner, the influence of the array manifold differential geometry on the accuracy of an azimuth-elevation direction-finding (DF) system employing a planar array of sensors.

In Section-2 the planar array manifold surface $a(\theta, \phi)$ is analyzed via a study of the differential geometry of its two sets of constant-parameter curves $a(\theta)$ and $a(\phi)$ referred to as $\theta$- and $\phi$-curves respectively. Two features of particular interest are the rate of change of arc length and the first curvature. The obtained results identify the manner in which the array configuration affects the shape of the array manifold. These are subsequently used in
Section-3 to evaluate expressions for the Cramer-Rao Lower bound on the variance of azimuth and elevation estimates in terms of:

- the local shape of the manifold $\theta$- and $\phi$-curves
- the signal-to-noise ratio
- the number of available snapshots

The scenarios investigated include the case of a single emitter as well as the critical case of two closely-spaced uncorrelated emitters. The derived expressions are extremely informative from an array design point of view, in the sense that they clearly describe the direct link between the array configuration and its DF performance.

2. MANIFOLDS OF PLANAR ARRAYS OF ISOTROPIC SENSORS

By a "planar array" we refer to a two dimensional distribution of sensors along a flat surface, e.g. the $(x, y)$-plane. The manifold of a planar array of $N$ isotropic sensors receiving narrow-band plane-wave signals from azimuth, $\theta$, and elevation, $\phi$, may be written as:

$$a(\theta, \phi) = \exp\left\{ - j \cdot \overrightarrow{r}^T k(\theta, \phi) \right\} = \exp\left\{ - j \pi \left( r_x \cos \theta + r_y \sin \theta \right) \cos \phi \right\}$$

$$\forall \theta, \phi \text{ such that } 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$$

where $\overrightarrow{x}^T = [x_1, \ldots, x_N]^T = [x_x, x_y, 0] \in \mathcal{R}^{N \times 3}$ is the matrix of sensor element locations (normalized to units of half-wavelengths) with respect to the $x$- and $y$-axes, $k(\theta, \phi) = \pi \left[ \cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi \right]^T \in \mathcal{R}^{3 \times 1}$ is the wavenumber vector pointing towards the emitter at $\theta, \phi$ ($\theta$ is measured anti-clockwise from the $x$-axis on the $xy$ plane, while $\phi$ is the angle made with the $xy$ plane). The limit on the values of $\phi$ disallows negative elevations and ensures that no trivial ambiguities exist.

Vector $\overrightarrow{A}(\theta, \phi)$ represents the relative phase of a signal arriving from direction $(\theta, \phi)$, at various sensors across the aperture of the array (measured with respect to a reference point). The dependence of $\overrightarrow{A}(\theta, \phi)$ on azimuth $\theta$ is a function of array shape and is given by $\overrightarrow{R}(\theta)$ while its dependence on elevation $\phi$ is simply cosinusoidal, irrespective of the array configuration. It can be readily seen that vector $\overrightarrow{R}(\theta)$ effectively represents the sensor locations if the array were to be projected upon a line along azimuth $\theta$.

2.1. Terminology regarding planar array geometry

The shape of a planar array is defined by vectors $x_x \in \mathcal{R}^{N \times 1}$ and $x_y \in \mathcal{R}^{N \times 1}$. It is advisable [3] to specify sensor locations with respect to the centroid of the array, or, alternatively, to choose the origin or phase reference to be at the array centroid. Then:

$$\text{sum}(x_x) = \text{sum}(x_y) = 0$$
**Definition** — A **symmetric** array may be defined as one where each sensor has a symmetric counterpart with respect to the array reference, i.e.

\[
\text{sum}(r_i^x) = \text{sum}(r_i^y) = 0 \quad \forall \ i = \text{odd} \quad (5)
\]

If the symmetry is with respect to the \(x\)-axis only, then \(\text{sum}(r_i^x) = 0\). Similarly if the symmetry is with respect to the \(y\)-axis only, then \(\text{sum}(r_i^y) = 0\).

Another class of planar geometries which are of particular interest are those for which vectors \(r_x\) and \(r_y\) are orthogonal:

\[
r_x^T r_y = 0 \quad (6)
\]

It is easy to verify that the symmetry of the array with respect to either the \(x\) or \(y\) axes (or both) is a sufficient — although not necessary — condition for the orthogonality of \(r_x\) and \(r_y\).

**Definition** — A **balanced symmetric** array may be defined as one for which the vectors \(r_x\) and \(r_y\) are orthogonal and have the same magnitude, that is :

\[
r_x^T r_y = 0 \quad \text{and} \quad |r_x| = |r_y| \quad (7)
\]

The equality of the norms implies that the sensors are in some way equally distributed in the \(x\) and \(y\) directions (although this may not be immediately apparent from a simple inspection of the array shape). Many popular structures including uniform-circular, X-shaped, Y-shaped, square-grid etc. fall into this category. As will be seen later, the manifolds of balanced-symmetric arrays have some unique features not shared by other array geometries.

Throughout the paper two specific array structures, actually located in the UK, will be used to illustrate the concepts involved. The geometries of these arrays are as follows :

i) A 24-element Y-shaped array with sensor locations of 
\[8,22,38,57,79,105,136,170\] 15 half-wavelengths \((f = 10 \text{ MHz})\) along each branch (branches separated by 120°). The array is balanced symmetric with 
\[|r_x| = |r_y| = 21.5875\].

ii) A 24-element uniform circular array with a radius of 75/15 half-wavelengths \((f = 10 \text{ MHz})\). This array is also balanced symmetric with 
\[|r_x| = |r_y| = 17.3205\].

### 2.2. Differential geometry of manifolds of planar arrays

The following facts may be readily deduced by inspection of Equation 3 with regards to the manifold of a planar array of \(N\) isotropic sensors :

- The array manifold is a 2-parameter surface lying on a hypersphere of radius \(\sqrt{N}\) embedded in complex \(N\)-space \(\mathbb{C}^N\). This is a result of the fact that
\[ |a(\theta, \phi)| = \sqrt{N} \] (8)

- The array manifold is a surface generated by the family of \( \phi \)-parameter curves which meet at the apex of the manifold at \( \phi = 90^\circ \).
- The \( \phi \)-curve manifold of a planar array, at azimuth \( \theta \), is identical to the manifold curve of a linear array with sensor locations \( \bar{R}(\theta) \) and is therefore shaped as a complex hyperhelix embedded in \( C^N \).

Figure 1 illustrates some of the above manifold features. Great care should be taken when interpreting pictorial representations of the manifold, since multidimensional complex geometry cannot always be shown consistently on a 2-dimensional diagram.

We may now proceed with the evaluation of the differential geometry of the \( \theta \)- and \( \phi \)-curves. The approach is similar to that presented in our earlier publication regarding the manifolds of linear arrays [2].

**FIGURE 1 :** Pictorial Representation of the \((\theta, \phi)\)-Manifold of a Planar Array

A study of the differential geometry [4, 5] of a general curve \( \bar{a}(p) \in C^N \) invariably begins with its length "s". A particularly useful feature is the rate of change of arc length defined as \( \dot{s}(p) = \frac{ds}{dp} = |\dot{a}(p)| \) where (\( \cdot \)) denotes differentiation with respect to parameter \( p \). The next step in the study of the geometry of a complex curve is to define a basis or set of coordinate vectors \( \mathbb{U}(s) = [u_1(s), \ldots, u_d(s)] \in C^{N \times d} \) at every point along the manifold curve in complex space \( C^N \), where \( d \leq 2N \). The shape of the curve may then be fully defined via a set of \( d-1 \) curvatures \( \{\kappa_1(s), \ldots, \kappa_{d-1}(s)\} \in \mathbb{R} \) which determine how the wide-sense orthonormal frame \( \mathbb{U}(s) \) rotates from point to point along the curve [3].

Fortunately, not all curvatures need to be evaluated if one is only interested in the local behaviour of a multidimensional curve. In fact it can be shown that [6], in the neighbourhood of a point \( s_o \), a curve \( \bar{a}(s) \in C^{N \times 1} \) has its main components along the unit tangent and normal vectors \( u_1(s_o) \) and \( u_2(s_o) \) respectively and can be interpreted as a circular arc of curvature \( \kappa_1(s_o) = \kappa_1(s_o) \sin \zeta(s_o) \).
where

\[
\begin{align*}
{\mathbf{u}_1} &= \frac{d}{ds'} = \text{Unit tangent vector}, \\
{\mathbf{u}_2} &= \frac{d}{ds'} / \kappa_1 = \text{Unit normal vector}, \\
\kappa_1 &= |\mathbf{u}_1'| = \text{First curvature} \\
\zeta(s_o) &= \arccos[\mathbf{u}_1^H(s_o)\mathbf{u}_2(s_o)]
\end{align*}
\]

with \( (\cdot) \) denoting differentiation with respect to arc length \( s \). The above circular approximation can be shown to be accurate over any segment \( a(s_o \pm \Delta s_i) \) of the array manifold so long as \( \Delta s_i \kappa_1(s_o) \ll 2 \), and is used to great effect later in Section-3.

In [6] the expressions for \( \dot{s}(p), \kappa_1(p) \) and \( \zeta(p) \) have been evaluated for a general \( p \)-parameter planar array manifold curve and the results are presented as follows:

\[
\dot{s}(p) = |\mathbf{A}| \\
\kappa_1(p) = |\mathbf{u}_1'| = \frac{1}{l_{\mathbf{r}}} \left| \mathbf{A}^2 + j \mathbf{P}_L \mathbf{A} \right| \\
\sin\zeta(p) = \sqrt{1 - |\mathbf{u}_1^H \mathbf{u}_2|^2} = \sqrt{1 - \frac{1}{s\kappa_1} \left[ \sum (\mathbf{A}^3) \right]^2}
\]

It is interesting to note that for arrays which are symmetric with respect to both the \( x \) and \( y \) axes \( \sum (\mathbf{A}^3) = 0 \) and hence \( \sin\zeta(p) = 1 \forall p \). Furthermore, recall that \( \widehat{\kappa}_1(p) \) (the curvature of the circular approximation) is related to the first curvature of the manifold by the product \( \widehat{\kappa}_1(p) = \kappa_1(p) \sin\zeta(p) \). Consequently, for the case of \( x-y \) symmetric arrays \( \mathbf{u}_1(p) \perp \mathbf{u}_2(p) \), and so the curvature of the circular approximation and the first curvature of the manifold are identical.

Expressions for the special cases of \( \theta \)- and \( \phi \)-curves are presented next.

### 2.2.1. Differential geometry of \( \theta \)-parameter curves

Since \( \mathbf{A}(\theta, \phi) = \pi \cos \phi \mathbf{R}(\theta) \) then \( \dot{\mathbf{A}} = \partial \mathbf{A}/\partial \theta = \pi \cos \phi \mathbf{R}_\theta(\theta) \) and so substitution into Equations 10-12 gives:

\[
\dot{s}_\theta(\theta) = \pi \cos \phi \left| \mathbf{R}_\theta \right| = \pi \cos \phi \left| -r_x \sin \theta + r_y \cos \theta \right| \\
\kappa_{1\theta}(\theta) = \frac{1}{l_{\mathbf{r}}} \left| \mathbf{R}_\theta^2 - \frac{j}{\pi \cos \phi} \mathbf{P}_L \mathbf{R} \right| \\
\sin\zeta_\theta(\theta) = \sqrt{1 - \frac{1}{s_{\theta}^2} \left[ \sum (\mathbf{R}_{\theta}^3) \right]^2}
\]

where \( \mathbf{R}_\theta = \mathbf{R}_\theta/|\mathbf{R}_\theta| \) and \( \mathbf{R}_\theta = \partial \mathbf{R}/\partial \theta \). Note that the differential geometry of the \( \theta \)-curves varies as a function of azimuth with a periodicity of at least 180°. Furthermore \( \kappa_{1\theta}(\theta) \) can be approximated, for large arrays at elevation not close to 90°, by \( |\mathbf{R}_{\theta}^2| \).

### 2.2.2. Differential geometry of \( \phi \) — parameter curves

Since \( \mathbf{A}(\theta, \phi) = \pi \cos \phi \mathbf{R}(\theta) \) then \( \dot{\mathbf{A}} = \partial \mathbf{A}/\partial \phi = -\pi \sin \phi \mathbf{R}(\theta) \) and so substitution into Equations 10-12 gives:
\[ \delta_\phi(\phi) = \pi \sin\phi \left| R \right| = \pi \sin\phi \left| \sum_x \cos \theta + \sum_y \sin \theta \right| \]  
(16)

\[ \kappa_1(\phi) = \frac{\left| \hat{A}^T \right|}{|A|} = \frac{\left| \hat{R} \right|}{|R|} = \left| \hat{R} \right| \]  
(17)

\[ \sin \zeta_\phi(\phi) = \sqrt{1 - \frac{1}{\kappa_0^2} \left[ \sum \left( \hat{R} \right) \right]^2} \]  
(18)

where \( \hat{R} = R / |R| \) is the normalized version of \( R \). Note that the differential geometry of the \( \phi \)-curves also varies as a function of azimuth with a periodicity of at least 180°.

The significance of Equations 13 to 18 will be thoroughly discussed in Section-3 in the context of the Cramer-Rao bound.

### 3. ACCURACY AND THE CRAMER RAO LOWER BOUND

The most popular bound in array processing is a well-known statistical result known as the Cramer-Rao lower bound (CRLB) [7]. The CRLB sets a lower limit on the error covariance matrix of any unbiased estimate, \( \hat{p} \), of the true parameter vector \( p \in R^M \) in the array model of Equation 1. In the case of an array of \( N \) sensors receiving \( M \) narrowband signals with additive sensor noise of power \( \sigma^2 \), and for a sufficiently large number of snapshots \( (L \gg 1) \), the expression for the deterministic CRLB has been shown to be as follows [8] :

\[ \text{CRB}[p] = \frac{\sigma^2}{\pi} \left( \text{Re}[H \otimes R_m^T] \right)^{-1} \in R^{M \times M} \]  
(19)

where \[ H = \hat{A}^H P_A \hat{A} \in C^{M \times M} \]  
\[ R_m = \mathcal{E}[m(t)m^H(t)] \in C^{M \times M} \]  
is the source covariance matrix

and the following assumptions are assumed to hold :

- \( N > M \) and any \( M+1 \) source position vectors are independent.
- Noise is a zero-mean, temporally white Gaussian process.
- Noise is spatially white from sensor to sensor; i.e. \( \mathcal{E} \{ n(t)n^H(t) \} = \sigma^2 I \).
- Bearing parameters other than \( p \) are known apriori.

In this Section we attempt to provide some insight into the nature of the CRLB and furthermore to clarify its relationship with the differential geometry of the array manifold (and hence the array geometry). This is achieved by concentrating attention on the special cases of one and two emitters.
3.1 Single Emitter CRLB in terms of manifold differential geometry

Assume that the array receives a single signal $m(t)$ of power $P = \mathcal{E}\{m(t)m^*(t)\}$ from bearing $p$. Then, since $R_m = P$ and $A = \hat{a}(p)$, Equation 19 implies that the CRLB may be expressed as:

$$\text{CRB}[p] = \frac{\sigma^2}{2LP} \frac{1}{\hat{a}^H \hat{P}_{\hat{a}} \hat{a}}$$

$$= \frac{\sigma^2}{2LP} \frac{1}{s^2(p)} \frac{1}{u_1^H \hat{P}_{\hat{a}} \hat{a} u_1}$$

recalling that $u_1 = \hat{a}(p)/s(p)$ and $|u_1| = 1$ ($u_1$ being the unit tangent vector). By choosing the phase reference to be at the array centroid, $u_1(p)$ can be made orthogonal to $\hat{a}(p)$ such that $\hat{P}_{\hat{a}} u_1 = u_1$. Hence Equation 20 may be further simplified as follows:

$$\text{CRB}[p] = \frac{\sigma^2}{2LP} \frac{1}{s^2(p)} = \frac{1}{2L \times \text{SNR} \ s^2(p)}$$

At this stage it is interesting to also consider the CRLB corresponding to the estimates of the manifold arc length parameter, $s$. Replacement of bearing parameter, $p$, with arc length, $s$, requires that the manifold derivative $\dot{a} = \partial \hat{a} / \partial p$, in Equation 20, is replaced by $\dot{a}' = \partial \hat{a} / \partial s = u_1$. Again since $\hat{P}_{\hat{a}} u_1 = u_1$, it follows that:

$$\text{CRB}[s] = \frac{\sigma^2}{2LP} \frac{1}{s^2} = \frac{1}{2L \times \text{SNR}}$$

In other words, the lower bound on the variance of the estimates of arc length parameter, $s$, is independent of the array geometry and depends only on the $L \times \text{SNR}$ product.

The next logical step is to see how Equation 21 can be written so as to provide information about the accuracy of $\theta$ and $\phi$ estimates in terms of the sensor locations. From the results of Section-2 regarding $\dot{\theta}$ and $\dot{\phi}$, and for $p = \theta$ and $p = \phi$, one may write:

$$\begin{align*}
\text{CRB}_\theta[\theta] &= \frac{1}{2L \times \text{SNR} \left(\pi \cos\phi \ |R_\theta(\theta)|\right)} \\
\text{CRB}_\phi[\phi] &= \frac{1}{2L \times \text{SNR} \left(\pi \sin\phi \ |R(\theta)|\right)}
\end{align*}$$

where as usual $R(\theta) = r_x \cos \theta + r_y \sin \theta$ and $R_\theta(\theta) = \frac{\partial R(\theta)}{\partial \theta}$.
3.1.1. DEDUCTIONS:

\( i \) For all planar array geometries:

\[
\begin{align*}
\text{CRB}_{\theta}[\theta] & \propto \frac{1}{\cos^2\phi} \quad \Rightarrow \, \theta \text{ estimates are more accurate for } \phi \to 0^\circ. \\
\text{CRB}_{\phi}[\phi] & \propto \frac{1}{\sin^2\phi} \quad \Rightarrow \, \phi \text{ estimates are more accurate for } \phi \to 90^\circ.
\end{align*}
\] (24)

The variations of \( \text{CRB}_{\theta}[\theta] \) and \( \text{CRB}_{\phi}[\phi] \) with respect to elevation \( \phi \) are both independent of array geometry, monotonic and 90° out of phase.

\( ii \) Since \( \mathbf{R}_{\theta}(\theta) = \mathbf{R}(\theta + \frac{\pi}{2}) \), consequently for all planar array geometries:

\[
\text{CRB}_{\theta}(\theta,\phi) = \text{CRB}_{\phi}\left(\theta + \frac{\pi}{2}, \frac{\pi}{2} - \phi\right)
\] (25)

In words, the CRLB for the \( \theta \)-estimates of an emitter at bearings \( (\theta,\phi) \) equals the CRLB for the \( \phi \)-estimates of a similar emitter at bearing \( (\theta + \frac{\pi}{2}, \frac{\pi}{2} - \phi) \). The variations of \( \text{CRB}_{\theta} \) and \( \text{CRB}_{\phi} \) with respect to azimuth \( \theta \) are both functions of array geometry, and are 90° out of phase.

\( iii \) For balanced-symmetric arrays \( |\mathbf{R}_{\theta}(\theta)|^2 = |\mathbf{R}(\theta)|^2 = |\mathbf{r}_x|^2 = (\text{independent of } \theta) \), therefore:

\[
\begin{align*}
\text{CRB}_{\theta}[\theta] &= \frac{1}{2L \times \text{SNR} \left(\frac{1}{\cos \phi} \left| \mathbf{r}_x \right| \right)^2} \quad \Rightarrow \, \theta\text{-accuracy is independent of } \theta \\
\text{CRB}_{\phi}[\phi] &= \frac{1}{2L \times \text{SNR} \left(\frac{1}{\sin \phi} \left| \mathbf{r}_x \right| \right)^2} \quad \Rightarrow \, \phi\text{-accuracy is independent of } \theta.
\end{align*}
\] (26)

3.1.2. EXAMPLE

Figures 2a-2d show values of \( \text{CRB}_{\theta}[\theta] \) and \( \text{CRB}_{\phi}[\phi] \) for a single emitter at bearing \( (\theta,\phi) \), as functions of \( \theta \) and \( \phi \). Equations 23 are evaluated for the 24-element Y-shaped and circular arrays introduced in Section-2. The figures are based on the assumption of \( L = 100 \) snapshots, and, a Signal-to-Noise ratio of 10dB. Both arrays are balanced-symmetric, and so the accuracy of the parameter estimates is independent of azimuth. On the other hand, while the accuracy of the parameter estimates varies dramatically with elevation, the performances of the two arrays remain unchanged in relative terms.
FIGURE 2 : CRLB for a Single Emitter with SNR × L = 1000
(SNR=10dB, L = 100)
N.B: Y-Array: - - - - Circular Array: - - - -
3.2. Two emitter CRLB in terms of manifold differential geometry

Expressions for the CRLB (on the variance of unbiased parameter estimates) become progressively more complicated with increasing numbers of emitters, \( M \), since the accuracy of the bearing estimates is not only a function of the additive sensor noise but also depend on the interactions between the various emitters.

Consider a multiple-emitter scenario involving two correlated emitters and \( M-2 \) uncorrelated emitters. It is easy to show that if the two correlated signals arrive from bearings \( p_1 \) and \( p_2 \) (corresponding to SPVs \( \bar{a}_1 \) and \( \bar{a}_2 \)), have powers \( P_1 \) and \( P_2 \), and a correlation coefficient \( \rho \), then Equation 19 for the CRLB may be written as :

\[
\text{CRB}[p_1] = \frac{1}{2L \times \text{SNR}_1} \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_1\|^2} \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_2\|^2} \frac{1 - R_{\hat{a}_1, \hat{a}_2}}{1 - R_{\hat{a}_1, \hat{a}_2}} \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_1\|^2 \|\mathbf{P}^H_{\mathbf{A}} \hat{a}_2\|^2}
\]

where \( \text{SNR}_1 = P_1 / \sigma^2 \). To make Equation 27 more tractable, consider the scenario where all \( M \) emitters are uncorrelated (emissions from independent sources). Setting \( \rho = 0 \) in Equation 27 we have

\[
\text{CRB}[p_1] = \frac{1}{2L \times \text{SNR}_1} \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_1\|^2}
\]

Making the substitution \( \hat{a}_{11} = \hat{a}_1(p_1) / \delta(p_1) \), one may simply write :

\[
\text{CRB}[p_1] = \frac{1}{2L \times \text{SNR}_1} \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_{11}\|^2} \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_{11}\|^2}
\]

for \( \rho = 0 \)

The dependence of the CRLB on the rate of change of manifold arc length, \( \delta(p) \), is again quite apparent. Further interpretation of Equation 29 in terms of array manifold differential geometry is possible when \( M = 2 \) (i.e. two-emitter scenario) and the two emitters are closely spaced at bearings \( p_1 \) and \( p_2 = p_1 + \Delta p \), corresponding to manifold arc lengths \( s_1 = s_o - \Delta s / 2 \) and \( s_2 = s_o + \Delta s / 2 \) respectively. Under such circumstances, circular approximation can be applied to a local neighbourhood of \( \hat{a}(s_o) \) in order to evaluate the term \( \frac{1}{\|\mathbf{P}^H_{\mathbf{A}} \hat{a}_{11}\|^2} \). As shown in the Appendix, this results in the following expression :

\[
\text{CRB}[p_1] = \frac{1}{L \times \text{SNR}_1} \frac{2}{\delta^2(p_1) (\Delta s)^2 \left( \frac{\kappa_2(p_o)}{\kappa_2(p) - \frac{1}{2}} \right)}
\]

for \( M = 2, \rho = 0 \), REF = CENTROID, \( \Delta s \leq \kappa_1(p) \leq \kappa_2(p) \leq 4 \)

where \( \Delta s = \delta(p_o) \Delta p \) and \( \kappa_1(p) = \kappa_1(p) \sin \zeta(p) \). Note that the bearing \( p_o \) corresponds to the point with arc length \( s_o \), and to a first-order approximation, equals \( (p_1 + p_2) / 2 \).

The technique deployed for the derivation of Equation 30 can also be directly applied to scenarios involving more than two closely spaced emitters. However, due to the proliferating number of manifold vector inner products, the expression for the CRB can become cumbersome. Nevertheless, it can be shown that so long as the additional manifold points \( s_3, \ldots \),
s_4, ..., s_M are not in the neighbourhood of s_o (i.e. in the neighbourhood of s_1 and s_2), the value of CRB[p_1] is primarily dominated by the presence of the source at bearing \( p_2 \).

### 3.2.1. DEDUCTIONS:

**i)** Since \( \Delta s = \hat{s}(p_o) - \Delta p \), Equation 30 indicates that the CRLB is inversely proportional to the second power of the emitters' angular separation,

\[
\text{CRB}[p_1] \propto \frac{1}{(\Delta p)^2}
\]

i.e. the accuracy of the system deteriorates rapidly as the emitters approach one another.

**ii)** The dependence of the CRLB on elevation \( \phi \) may be deduced by recalling from Section-2 that \( \hat{s}_0(\theta) \propto \cos \phi \) and \( \hat{s}_\phi(\phi) \propto \sin \phi \). Also since the \( \phi \)-curves are hyperhelical, \( \hat{\kappa}_{1\phi}(\phi) \) is independent of \( \phi \). Furthermore, for large arrays, \( \hat{\kappa}_{1\phi}(\phi) \) is independent of \( \phi \) so long as elevation is not close to 90°. The combination of these results, along with Equation 30 reveal the following dependencies on elevation \( \phi \):

- For emitters with the same elevation angle, i.e. \((\theta_1, \phi_1)\) and \((\theta_2, \phi_2)\),

\[
\text{CRB}[\theta_1] \propto \frac{1}{\hat{s}_0'(\theta_1) \hat{s}_\phi'(\phi_o) \hat{\kappa}_{1\phi}'(\phi_o)} \propto \frac{1}{\cos^2 \phi_o}
\]

where \( \phi_o = \frac{\phi_1 + \phi_2}{2} \)

- For emitters with the same azimuth angle, i.e. \((\theta, \phi_1)\) and \((\theta, \phi_2)\),

\[
\text{CRB}[\phi_1] \propto \frac{1}{\hat{s}_0'(\phi_1) \hat{s}_\phi'(\phi_o) \hat{\kappa}_{1\phi}'(\phi_o)} \propto \frac{1}{\sin^2 \phi_1 \sin^2 \phi_o}
\]

where \( \phi_o = \frac{\phi_1 + \phi_2}{2} \)

Consequently, the variations of \( \text{CRB}[\theta_1] \) and \( \text{CRB}[\phi_1] \) with respect to \( \phi \) are independent of array geometry and 90° out of phase.

**iii)** The dependence of the CRLB on azimuth \( \theta \) is a rather complex function of the array geometry. However, it is known from the results of Section-2 that \( \hat{s}_\theta(\theta, \phi) = \hat{s}_\theta(\theta + \frac{\pi}{2}, \phi) \) and that \( \hat{\kappa}_{1\theta}(\theta, \phi) \approx \hat{\kappa}_{1\phi}(\theta + \frac{\pi}{2}, \phi) = (\text{independent of } \phi) \). Incorporating these results into Equation 30, and on the grounds that for closely spaced emitters \( \hat{s}_\theta(\theta_1) \approx \hat{s}_\theta(\theta_o) \) and \( \hat{s}_\phi(\phi_1) \approx \hat{s}_\phi(\phi_o) \), one may deduce that:

\[
\text{if } \Delta \theta = \Delta \phi \quad \text{then} \quad \text{CRB}[\theta]\text{[}\phi\text{]} \approx \text{CRB}[\theta]\text{[}\frac{\pi}{2} - \phi \mid \theta + \frac{\pi}{2}\text{]}
\]

In other words, the CRLB on the azimuth estimates of two emitters equally distributed about azimuth \( \theta \) and with common elevations \( \phi \), is equal to the CRLB on the elevation estimates of two similar emitters equally distributed about elevation
and with common azimuth \( \theta + \frac{\pi}{2} \). This is somewhat similar to the result derived for the single-emitter case (see Equation 25) in that the variations of \( \text{CRB}_0[\theta_1] \) and \( \text{CRB}_\varphi[\phi_1] \) with respect to \( \theta \) are 90° out of phase.

### 3.2.2. Example

In Figure 3 the value of \( \text{CRB}_0[\theta_1] \), as defined in [8], is compared to that given by Equation 30 for two unit-power emitters located at bearings \( \theta_1 = 20° \) and \( \theta_2 = \theta_1 + \Delta \theta \), and both at a common elevation of \( \phi = 20° \). The expressions are evaluated for the 24-element Y-shaped array, assuming the availability of \( L = 100 \) snapshots and a signal-to-noise ratio of 10dB (for each emitter).

![FIGURE 3: Estimation of the CRB based on Equation 30](image)

As can be seen, for small emitter separations, Equation 30 provides an excellent estimate of the CRLB. However for increasing separations the circular approximation breaks down and Equation 30 is no longer valid. Note that for large emitter separations, the CRLB settles down to the value corresponding to a single emitter.

Figures 4(a-b) show values of \( \text{CRB}_0[\theta_1] \) (Equation 30) as a function of \( \theta \) and \( \phi \), in the case of two unit-power emitters of common elevation \( \phi \) but separated by \( \Delta \theta = 1° \) about azimuth \( \theta \). The bound is evaluated for the 24-element Y-shaped and circular arrays assuming the availability of \( L = 100 \) snapshots and signal-to-noise ratio of 10dB (for each emitter). In Figures 4(c-d) the exercise is repeated for \( \text{CRB}_\varphi[\phi_1] \) with emitters of common azimuth \( \theta \) but separated by \( \Delta \phi = 1° \) about elevation \( \phi \).

While the circular array again exhibits uniform accuracy for all azimuths, the Y-shaped array shows fluctuations with a period of 60° due to its special shape. Again, the
performances of the two arrays remain unchanged in relative terms as a function of elevation (true for \( \text{CRB}_0[\theta] \) for elevations not too close to 90°) as can be readily confirmed from the graphs.

**FIGURE 4 : CRLB for Two Emitters with \( \text{SNR} \times L = 1000 \) (\( \text{SNR}=10\text{dB}, L = 100 \))

N.B: Y-Array: --- Circular Array: - - -
4. CONCLUSIONS

In this paper the deterministic Cramer-Rao bound on the error of bearing estimates, provided by an array of sensors in a direction-finding system, has been investigated and its behaviour in the case of planar array geometries has been studied in some detail. Also a novel expression for the CRLB has been derived in terms of array manifold differential geometry for the case of two closely spaced emitters. This provides some considerable insight into the relationship between the array configuration and its DF performance.

It has been shown that while the variations of the CRLB with respect to $\theta$ are strongly dictated by the structure of the planar array, the variations as a function of $\phi$ are independent of the array geometry and are a consequence of the change in effective planar aperture at different elevations.

5. REFERENCES


APPENDIX:
Cramer-Rao Bound for Two Uncorrelated Emitters in Terms of Manifold Differential Geometry

Let $A = [\tilde{a}_1, A_r]$ where $A_r = [a_2, \ldots, a_M]$. Then the term $u_{11}^H \mathbf{P}_A^{-1} u_{11}$ of Equation 29 can be rewritten as follows:

\[
 u_{11}^H \mathbf{P}_A^{-1} u_{11} = u_{11}^H \left\{ \mathbf{P}_A^{-1} - \mathbf{P}_A^{-1} A_r \left( A_r A_r^H \mathbf{P}_A^{-1} A_r \right)^{-1} A_r A_r^H \mathbf{P}_A^{-1} \right\} u_{11}
\]

\[
 = u_{11}^H \mathbf{I} - u_{11}^H A_r \left( A_r A_r^H \mathbf{P}_A^{-1} A_r \right)^{-1} A_r^H u_{11}
\]

\[
 = 1 - u_{11}^H A_r \left( A_r^H \mathbf{P}_A^{-1} A_r \right)^{-1} A_r^H u_{11}
\]

Substituting the above equation back into Equation 29 and then expressing the columns of the matrix $A_r$ as $\tilde{a}_i = a_1 + \Delta \tilde{a}_i$ (for $i = 2, \ldots, M$), the CRLB on the bearing $p_1$ can be expressed as a function of the array manifold parameters.

However, it is informative to evaluate the CRLB in critical circumstances where there are two emitters ($M = 2$) which are closely spaced at bearings $p_1$ and $p_2 = p_1 + \Delta p$, corresponding to arc lengths $s_1 = s_o - \Delta s/2$ and $s_2 = s_o + \Delta s/2$ respectively.

In this case the matrix $A_r$ becomes the manifold vector $\tilde{a}_2$ and Equation 35 is simplified as follows:

\[
 u_{11}^H \mathbf{P}_A^{-1} u_{11} = 1 - \frac{N |\tilde{a}_2|^2}{N^2 - |\tilde{a}_1|^2 |\tilde{a}_2|^2} \tag{36}
\]

The physical proximity of the arriving signals allows the use of local differential geometry for the evaluation of Equation 36 (or, of Equation 35 for the general case of $M > 2$).

a) Evaluation of the term $|u_{11}^H a_2| = |u_1(s_1)^H a(s_2)|$.

Clearly:

\[
 u_1(s_1)^H a(s_2) = u_1(s_1) \left( a(s_1) + \Delta a \right) = u_1(s_1) \Delta a \tag{37}
\]

As established in Section (2), the manifold in the neighbourhood of $s_o$ may be interpreted as a circular arc of radius $1/\hat{\kappa}_1(s_o)$ as depicted in Figure 5.

In such circumstances the inner product may be written as:

\[
 u_1^H(s_1) \Delta a \simeq |\Delta a| \cos \Delta \psi \]

\[
 \simeq |\Delta a| \sqrt{1 - \sin^2 \Delta \psi} \]

\[
 \simeq |\Delta a| \sqrt{1 - \frac{1}{4} |\Delta a|^2 \hat{\kappa}_1^2(s_o)}
\]

or:

\[
 u_1^H(s_1) a(s_2) \simeq |\Delta a| \sqrt{1 - \frac{1}{4} |\Delta a|^2 \hat{\kappa}_1^2(s_o)} \tag{38}
\]
b) Evaluation of the term $a_1^H a_2 = a_1^H(s_1) a(s_2)$.

Clearly:

$$a_1^H(s_1) a(s_2) = a_1^H(s_1) (a(s_1) + \Delta \hat{a}) = N + a_1^H(s_1) \Delta \hat{a}$$

(39)

The array manifold has constant norm and hence lies on the surface of a hypersphere. Consequently, for sufficiently small $\Delta s$, vectors $a(s_n)$ and $\Delta \hat{a}$ are strictly orthogonal\footnote{In other words, with the reference taken at the array centroid, $\lim_{\Delta s \to 0} \left\{ a_1^H(s_n) \frac{\Delta \hat{a}}{\Delta s} \right\} = a_1^H(s_n) a_1(s_n) = 0$} as indicated in Figure 6.

Using simple trigonometry:

$$a_1^H(s_1) \Delta \hat{a} \approx \sqrt{N} |\Delta \hat{a}| \cos\left(\frac{\pi}{2} + \delta\right)$$

$$\approx -\sqrt{N} |\Delta \hat{a}| \sin(\delta)$$

$$\approx -\sqrt{N} |\Delta \hat{a}| \frac{|\Delta \hat{a}|}{2\sqrt{N}} = -\frac{1}{2} |\Delta \hat{a}|^2$$

Then:

$$a_1^H(s_1) a(s_2) \approx N - \frac{1}{2} |\Delta \hat{a}|^2$$

(40)
A slight variation, with $\Delta s$ replacing $|\Delta \mathbf{a}|$, may be obtained via a second-order Taylor expansion of $\mathbf{a}(s_2)$ about $s_1$. However simulations indicate that Equation 40 provides a more accurate approximation.

Substituting (38) and (40) back into (36) and then using Equation 29:

$$\text{CRB}[p_1] = \frac{1}{2L \times \text{SNR}_1 \delta^2(p_1)} \left\{ 1 - \frac{N |\Delta \mathbf{a}|^2 \left( 1 - \frac{1}{4} |\Delta \mathbf{a}|^2 \hat{\kappa}_1^2(s_0) \right)}{N^2 - \left( N - \frac{1}{2} |\Delta \mathbf{a}|^2 \right)^2} \right\}^{-1}$$

$$= \frac{1}{2L \times \text{SNR}_1 \delta^2(p_1)} \left\{ 1 - \frac{N \left( 1 - \frac{1}{4} |\Delta \mathbf{a}|^2 \hat{\kappa}_1^2(s_0) \right)}{N - \frac{1}{4} |\Delta \mathbf{a}|^2} \right\}^{-1}$$

$$= \frac{1}{2L \times \text{SNR}_1 \delta^2(p_1)} \frac{4N - |\Delta \mathbf{a}|^2}{|\Delta \mathbf{a}|^2 \left( N \hat{\kappa}_1^2(s_0) - 1 \right)}$$

So finally:

$$\text{CRB}[p_1] \simeq \frac{1}{L \times \text{SNR}_1 \frac{2}{\delta^2(p_1) \Delta s^2 \left( \hat{\kappa}_1^2(s_0) - \frac{1}{N} \right)}}$$  \hspace{1cm} (41)

where $|\Delta \mathbf{a}| \simeq \Delta s = \Delta p \delta(p_0)$ and it has been assumed that $4N \gg |\Delta \mathbf{a}|^2$. \hfill \Box