Array Shape Calibration using a Single Multi-Carrier Pilot

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Abstract—In this paper, a novel single pilot array shape calibration algorithm is proposed for an arbitrary planar array. The method requires a single multi-carrier pilot operating at a known location with respect to the arbitrary array reference point. Typically two (minimum) or more sources are required to calibrate the shape of a planar array. However, by exploiting the difference in the array response model when the source operates in the "near-far" and "far" field of the array, it is shown how this can be reduced to just one pilot. Simulation results exhibiting the performance of the proposed method are also presented.

Notation

- $A$, $a$ Scalar
- $A$, $a$ Column vector
- $\hat{A}$, $\hat{a}$ Matrix
- $(\cdot)^T$ Transpose
- $(\cdot)^H$ Hermitian Transpose
- $||\cdot||_2$ Norm of a vector
- $\odot$ Hadamard product
- $\exp(A)$ Element by element exponential of vector $A$
- $\text{row}_i(A)$ $i^{th}$ row of matrix $A$
- $a^b$ Element by element power
- $\mathbf{1}_N$ Column vector of $N$ ones
- $\mathbf{I}_N$ $N \times N$ Identity matrix
- $\mathcal{L}[A]$ Linear subspace spanned by the columns of $A$
- $\mathcal{E}\{\cdot\}$ Expectation operator
- $\angle(a)$ Element by element angle of the complex vector $a$
- $||a||_2$ Element by element floor of vector $a$
- $\mathbb{R}$ Set of real numbers
- $\mathbb{C}$ Set of complex numbers

I. INTRODUCTION

The presence of uncertainties in an array system causes rapid degradation in the detection, resolution and estimation performance of array processing algorithms. Array uncertainties include gain, phase, sensor location and sensor mutual coupling effects [1], [2]. Sensor location uncertainties are particularly degrading to the performance of an array system since they introduce a direction-dependant uncertainty. As a result they are also less trivial to estimate. This problem is illustrated in Figure 1 which shows the result of a superresolution direction finding algorithm (the estimation problem) with and without sensor location uncertainties for a Uniform Circular Array (UCA) of $N = 7$ sensors with half wavelength inter-sensor spacing. The location uncertainties imposed on each sensor are within the plane of the array and have an average of 0.0731 in half wavelength units. The MUSIC algorithm is employed to estimate the Direction-of-Arrival (DOA) of three uncorrelated sources in the "far" field of the array at $(\text{azimuth}, \text{elevation})$ directions $(30^\circ, 0^\circ)$, $(35^\circ, 0^\circ)$ and $(120^\circ, 0^\circ)$ with $L = 1000$ snapshots under an SNR of 20dB. The presence of uncertainties clearly causes significant degradation to the performance of the algorithm. Similar issues can be observed in the reception (beamforming) problem. Hence, it is imperative to detect, estimate and compensate for errors/uncertainties in the locations of the array elements.

The two main approaches to array calibration are pilot and self calibration. In pilot calibration, sources with some known...
parameters are used to estimate array uncertainties analytically by exploiting the mathematical model of the array response. For example, in [3], three or more "far" field pilot sources are used to estimate gain, phase, sensor mutual coupling effects and array shape uncertainties. Furthermore, in [4], it is shown how a single moving pilot source can be used to estimate the array shape. These methods tend to be highly accurate but are based on the assumption that certain pilot source parameters are known. Furthermore, they require the pilot sources to be placed at different locations which may be undesirable. In [5], a method for the calibration of an acoustic array based on time difference of arrival is proposed where only an approximate position of several sources are required. However, time difference of arrival methods are known to suffer from bandwidth limitations and multipath effects. In self-calibration (e.g. [6], [7], [8]), unknown source parameters and array uncertainties are estimated simultaneously. Since there are many more unknowns than equations, a cost function is typically optimised to solve this problem, which in general is highly nonlinear making traditional gradient based approaches unsuitable. Thus a suitable optimisation algorithm must be chosen which is prone to rapid and robust convergence.

This paper is concerned with pilot based array shape calibration in the case where there is a single pilot source available in a fixed location. Since array shape uncertainties produce direction-dependant uncertainties, they are considered to be the most complex to estimate, requiring the most overheads and producing the largest degradation in performance of the array system. In particular, a novel approach is proposed which uses only a single stationary multi-carrier pilot source at a known location. The pilot must transmit on at least two subtly chosen carrier frequencies.

The remainder of this paper is organised as follows. In Section II, the plane-wave and spherical-wave propagation models and the conditions under which they are valid are presented. Furthermore, the array shape calibration problem is formulated. In Section III, the proposed calibration algorithm is described. Next, in Section IV, the performance of the algorithm is demonstrated with some representative examples related to the DOA estimation and beamforming problems. Finally, in Section V, the paper is concluded.

II. PROBLEM FORMULATION

Consider an array of \( N \) sensors operating in the presence of a single source located at range \( \rho_1 \), azimuth angle \( \theta \) and elevation angle \( \phi \) with respect to an arbitrary array reference point which without any loss of generality is assumed to be at the first sensor. The signal vector \( \mathbf{s}(t) \in \mathbb{C}^N \) received by the array can be modelled as

\[
\mathbf{s}(t) = \mathbf{S}m(t) + \mathbf{n}(t) \quad (1)
\]

In Equation 1, the vector \( \mathbf{s}(t) \triangleq \mathbf{S}(\theta, \phi, \rho_1, \mathbf{r}, F_c) \) represents the \( N \)-dimensional array manifold vector (array response vector), where \( F_c \) is the carrier frequency of the source and \( \mathbf{r} \) is a \( 3 \times N \) matrix containing the Cartesian coordinates of the sensors in the array (in metres) with respect to the array reference point of the form

\[
\mathbf{r} = [\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z]^T \in \mathbb{R}^{3 \times N} \quad (2)
\]

In addition, \( m(t) \) is the baseband message signal transmitted by the source. Furthermore, \( \mathbf{n}(t) \) is the complex \( N \)-dimensional vector of the noise at the array elements. Without any loss of generality, it can be assumed that the noise is complex zero mean additive white Gaussian noise with covariance matrix

\[
\mathbf{R}_nn = \sigma_n^2 \mathbf{I}_N \quad (3)
\]

where \( \sigma_n^2 \) denotes the noise power. The array manifold vector follows a spherical-wave propagation model. Assuming omnidirectional sensors with the first element as the reference point, this is defined by

\[
\mathbf{s} = (\rho_1^2 \cdot \mathbf{L}^{-a}) \odot \exp \left( -\frac{2\pi F_c}{c} (\rho_1 \cdot \mathbf{1}_N - \mathbf{r}) \right) \in \mathbb{C}^N \quad (4)
\]

where

\[
\begin{align*}
\alpha & : \text{path loss exponent} \\
\epsilon & : \text{signal propagation speed}
\end{align*}
\]

In Equation 4, the vector

\[
\mathbf{r} = [\rho_1, \rho_2, \ldots, \rho_N]^T \in \mathbb{R}^N
\]

where its \( i \)-th element \( \rho_i \) describes the range from the source to the \( i \)-th sensor in the array (in meters). Collectively, \( \mathbf{r} \) can be defined mathematically as

\[
\mathbf{r} = \rho(\theta, \phi, t, \mathbf{r}, F_c)
\]

where

\[
\mathbf{r} = \sqrt{\rho_1^2 \cdot \mathbf{1}_N + \mathbf{r}_x^2 + \mathbf{r}_y^2 + \mathbf{r}_z^2 - \rho_1^2 \mathbf{r}^T \mathbf{k}(\theta, \phi)} \quad (6)
\]

where \( \mathbf{k}(\theta, \phi) \) is the wavenumber vector and is defined by the following equation

\[
\mathbf{k}(\theta, \phi) = \frac{2\pi F_c}{c} [\cos(\theta) \cos(\phi), \sin(\theta) \cos(\phi), \sin(\phi)]^T
\]

However, when the source lies in the "far" field of the array, it is widely accepted that the array manifold vector approximately follows a plane-wave propagation model. Again, assuming omnidirectional sensors, this is defined by

\[
\mathbf{s} = \exp \left( -j \mathbf{r}^T \mathbf{k}(\theta, \phi) \right) \in \mathbb{C}^N
\]

Note that as \( \rho_1 \to \infty \) (large, i.e. "far field"), the limit of Equation 4 is Equation 8. A source can be considered to be in the "near-far" field of the array when

\[
\rho_1 > \frac{D^2 F_c}{c} \quad \text{in meters} \quad (9)
\]

where \( D \) is the array aperture in metres. Equation 9 implies that by suitably selecting/changing the carrier frequency of the source, its fixed physical location can change from being in the "near-far" field of the array to the "far" field. Hence, Equation 4 can be used to describe the array response at one frequency and Equation 8 can be used with good approximation in
another. This forms the underlying concept in which the proposed calibration algorithm is based.

Consider that the array contains sensor location uncertainties described by the matrix \( \tilde{r} \), where the true locations of the sensors described by the matrix \( r \) can be expressed in terms of the nominal locations, \( \tilde{r} \), by

\[
\mathbf{r} = \mathbf{\tilde{r}} + \mathbf{\tilde{r}}
\]

(10)

Without loss of generality, it is assumed that sensor 1 is at a known location which is also taken to be the default array reference point. The presence of location uncertainties causes the true manifold vector \( S \) to deviate from its nominal value \( \tilde{S} \). Thus it is assumed that the manifold vector \( \tilde{S} \) describes the array response rather than the true manifold vector \( S \). This leads, for instance, to the performance decrease observed in Figure 1. The array shape calibration problem is concerned with estimating the matrix \( r \).

The covariance matrix of the data received from an array with respect to the array reference point in the presence of a pilot source transmitting at power \( P_s \) from a known location can be modelled as:

\[
\mathbb{R}_{xx} = P_s S \mathbb{S} \mathbb{S}^H + \mathbb{R}_{nn}
\]

(11)

where \( \mathbb{R}_{nn} \) is the noise covariance matrix as described in Equation 3. Performing eigenvector decomposition on \( \mathbb{R}_{xx} \) it is well known that the eigenvector corresponding to the principal eigenvalue \( F \mathbb{S} \) spans the same linear subspace as the manifold vector \( \mathbb{S} \), i.e.:

\[
\mathbb{S} \in \mathcal{E} \{ \mathbb{E} \}
\]

(12)

\[
\mathbb{E} = \text{const} \cdot \mathbb{E}
\]

(13)

Where\(^1\) \( \text{const} = \left\{ \begin{array}{ll} \sqrt{N} & \text{plane-wave propagation} \\ \| \rho_1 \cdot \mathbb{E} \| & \text{spherical-wave propagation} \end{array} \right. \)

III. ARRAY SHAPE CALIBRATION APPROACH

Consider an uncalibrated planar array of \( N \) sensors with array shape uncertainties which operate in the presence of one pilot source located at a known range \( \rho_1 \) and (azimuth, elevation) = (\( \phi \), \( \theta \)) with respect to the default array reference point at sensor 1 of the array. Without any loss of generality, assume elevation angle \( \phi = 0^\circ \). Assume the pilot source transmits a baseband message \( m(t) \) on a carrier frequency \( F_{c1} \) placement it in the "near-far" field of the array following Equation 8. Hence, the array response follows a plane-wave propagation of the form

\[
\mathbb{S} = \exp \left( -j \frac{2 \pi F_{c1}}{c} (x \cos (\theta) + y \sin (\theta)) \right)
\]

(14)

The element by element angle of the signal eigenvector \( \angle (\mathbb{E}) \) corresponds to

\[
\angle (\mathbb{E}) + 2 \pi K = - \frac{2 \pi F_{c1}}{c} (x \cos (\theta) + y \sin (\theta))
\]

(15)

where

\[
K = \left[ \begin{array}{c} K_1, \ K_2, \ \cdots \ \ K_N \end{array} \right]^T
\]

(16)

which is a vector of integers due to \( \angle (\mathbb{E}) \) having a range \([0, 2\pi]\), with \( K_1 = 0 \). This vector \( K \) can be estimated using the nominal values of the sensor locations assuming the location error is less than \( \lambda \), i.e.

\[
\mathbb{K} = \left[ \begin{array}{c} - \frac{F_{c1}}{c} (x \cos (\theta) + y \sin (\theta)) \end{array} \right]
\]

(17)

Equation 15 can be re-written as

\[
- \frac{c}{2 \pi F_{c1}} (\angle (\mathbb{E}) + 2 \pi K) = x \cos (\theta) + y \sin (\theta)
\]

(18)

Equation 18 is a set of \( N \) linear equations of \( 2N \) unknowns \( x \) and \( y \).

Next consider that the source transmits the baseband message \( m(t) \) on a carrier frequency \( F_{c2} \) (where \( F_{c2} > F_{c1} \)) either at the same (followed by digital filtering) or different time to transmission on \( F_{c1} \), which now places it in the "near-far" field of the array. Hence, the array response now follows Equation 4. Under spherical-wave propagation, the array undergoes changes in gain as well as phase as a function of the array and source locations. Consider \( L \) snapshots of data received from the \( N \) sensor array. The signals received over this observation interval can be expressed as the \( N \times L \) matrix \( \mathbb{X} \) where:

\[
\mathbb{X} = \left[ \mathbb{x}(1), \ \mathbb{x}(2), \ \cdots, \ \mathbb{x}(L) \right] \in \mathbb{C}^{N \times L}
\]

(19)

This follows the signal model presented in Equation 1. For the purposes of this algorithm, the array reference point for the matrix \( \mathbb{X} \) can be set to the \( i^{th} \) sensor by dividing the signals received at each sensor by the signals received at the \( i^{th} \) sensor. By rotating the array reference point to be at each of the sensors in the array in turn, \( N \) covariance matrices can be constructed as:

\[
\mathbb{R}_i = \frac{1}{L} \sum_{i=1}^{L} \mathbb{x}_i \mathbb{x}_i^H \in \mathbb{C}^{N \times N}
\]

(20)

where:

\[
\mathbb{x}_i = \mathbb{X} \odot (\mathbb{I}_N \cdot \text{row}_i (\mathbb{X}))
\]

(21)

In [9], it is proved that:

\[
\left( \frac{\lambda_i}{\lambda_1} \right)^{\frac{1}{N}} = \frac{\rho_i}{\rho_1}
\]

(22)

where \( \lambda_i \) corresponds to the largest eigenvalue minus the average of the remaining \( N-1 \) eigenvalues of the covariance matrix \( \mathbb{R}_i \) (when the reference point is at the \( i^{th} \) sensor of the array). Furthermore, \( \lambda_1 \) corresponds to the largest eigenvalue
minus the average of the remaining \( N - 1 \) eigenvalues of \( \mathbb{R}_1 \) (when the reference point is at the global reference point - taken to be the 1st sensor with no location uncertainties associated with it). Furthermore, \( \rho_1 \) denotes the range between the source and the \( i \)th sensor. Hence, the range between the source and each of the array sensors is given by the vector

\[
\Delta = \begin{bmatrix} \lambda_1, \lambda_2, \ldots, \lambda_N \end{bmatrix}^T
\]

as:

\[
\rho = \rho_1 \left( \frac{1}{\lambda_1} \Delta \right)^\frac{1}{n}
\]  

(23)

Following this, using Equation 23 and knowing the range from the pilot to the array reference point \( \rho_1 \), the range between the source and each of the array sensors \( \rho \) can be estimated. Following this, using Equation 6, a second set of \( N \) linear equations with \( 2N \) unknowns, \( r_x \) and \( r_y \), can be found as

\[
\rho^2 - \rho_1^2 \cdot \frac{1}{n} = \rho_1^2 + \Delta^2 - 2\rho_1 \left( r_x \cos(\theta) - r_y \sin(\theta) \right)
\]  

(24)

Finding \( u \) and \( v \) using the received array signals in the presence of the pilot source, it can be proved by combining Equations 18 and 24 that:

\[
r_y = \frac{u \sin(\theta) \pm \sqrt{(2\rho_1 - u^2 + v) \cos(\theta)}}{\cos(\theta) - \tan(\theta)} r_y
\]  

(25)

\[
r_x = \frac{1}{\cos(\theta)} u - \tan(\theta) r_y
\]  

(26)

This provides two sets of solutions, i.e. two points of intersection which is intuitive since a spherical-wave will intersect with a plane-wave at two points. However, knowing the nominal array locations, the true solution can be chosen as the closest point to this nominal location. Note that this method requires the transmitter to operate at just two carrier frequencies - one causing the source to be in the "near-far" field of the array and one in the "far". However, the approach could be extended to operate at more than two carriers. Multiple carriers will provide solutions that intersect at the same common points. These points can be averaged to allow a more accurate approximation of the sensor location uncertainties to be made.

The proposed algorithm can be presented as a series of steps as follows:

**STEP-1** For the transmission at the lower carrier frequency \((F_{c_1}, \text{plane-wave propagation})\), estimate the received signal covariance matrix \( \mathbb{R}_{xx} \) when the array reference point is at sensor 1 using Equation 11 and find the principle eigenvector \( \mathbf{F}_{c_1} \).

**STEP-2** Estimate the associated \( K \) using Equation 17 then compute the vector \( \mathbf{u} \) defined in Equation 18.

**STEP-3** For the transmission at the higher carrier frequency \((F_{c_2}, \text{spherical-wave propagation})\), construct the received signal covariance matrices \( \mathbb{R}_i \) for \( i = 1, 2, \ldots, N \) by rotating the array reference point using Equations 20 and 21 and for each case find the principle eigenvalue \( \lambda_i \).

**STEP-4** Using \( \rho_1 \) estimate the range \( \rho \) between the pilot and each of the array sensors using Equation 23.

**STEP-5** Using \( \rho \), compute the vector \( \mathbf{v} \) associated with the pilot source defined in Equation 24.

**STEP-6** Calculate the sensor locations \([r_x, r_y]\) using Equations 25 and 26.

Note that in the case of \( M \) spherical-wave transmissions, steps 3, 4 and 5 should be repeated for each and then used in step 6 to produce \( M \) estimates of \([r_x, r_y]\) which can be averaged to reduce finite averaging effects. These effects are present due to the use of \( L \) (finite) snapshots in the estimation of \( u \) and \( v \).

**IV. REPRESENTATIVE EXAMPLES**

The proposed method significantly improves the performance of an array system by removing the array location uncertainties. It has been tested under a number of different planar array geometries with varying numbers of sensors and under various degrees of sensor location uncertainty and performed satisfactorily. For example, consider a Uniform Circular Array of \( N = 7 \) sensors operating in the presence of a single pilot source \( \rho_1 = 6 \) metres (\( m \)) from the first sensor (taken to be the array reference point) at an azimuth of \( \theta = 60^\circ \) and an elevation of \( \phi = 0^\circ \). The pilot source transmits at carrier frequencies of \( F_{c_1} = 500\text{MHz} \) and \( F_{c_2} = 2.45\text{GHz} \). Furthermore, the array has a nominal half unit wavelength sensor spacing between adjacent sensors with respect to \( F_{c_1} \) giving an array aperture of \( D = 0.6741\text{m} \). Hence, the boundary between the "near-far" and "far" fields for each of the carrier frequencies lies at 1.5147\( m \) for \( F_{c_1} \) and 7.4250\( m \) for \( F_{c_2} \). Therefore, the pilot source at \( \rho_1 = 6\text{m} \) allows a plane-wave propagation at \( F_{c_1} \) and a spherical-wave propagation at \( F_{c_2} \). Figure 2 and Table 1 summarise the results after implementing the proposed algorithm when the pilot is observed for \( L = 1000 \) snapshots under an SNR of 20dB.

The proposed calibration technique was tested with 3 uncorrelated sources operating in the "far" field of the array at \( F_{c_1} \) for the setup illustrated in Figure 2 at (azimuth, elevation) = (30\(^\circ\), 0\(^\circ\)), (35\(^\circ\), 0\(^\circ\)) and (120\(^\circ\), 0\(^\circ\)) with \( L = 1000 \) snapshots under an SNR of 20dB. The MUSIC algorithm is employed before and after the proposed calibration approach to illustrate the improved capabilities in solving the estimation problem as a result of the calibration approach. Results in Figure 3 show a significant improvement in the performance of the algorithm following calibration.

Next consider that for the setup illustrated in Figure 2, the source at \( \theta = 30^\circ \) is the desired source and the sources at 35\(^\circ\)
and 120° are co-channel interferences that must be rejected. A super-resolution beamformer which is designed to maximise Signal to Interference Ratio (SIR) is employed to achieve this. The array pattern before and after the proposed calibration approach is illustrated in Figure 4. Before calibration it is clear that the beamformer is unable to reject the 35° and 120° interferences. After calibration the nulls are far sharper, deeper and placed accurately in the interference directions showing a significant improvement in the SNIR performance of the beamformer. Now consider that the source at (azimuth, elevation) = (120°, 0°) is the desired source and the sources at 30° and 35° are co-channel interferences with the setup illustrated in Figure 2 under SNR of 20dB and L = 1000 snapshots. The Wiener-Hopf beamformer is employed before and after the proposed calibration approach producing the array patterns illustrated in Figure 5. This beamformer is designed to maximise Signal to Noise Interference Ratio (SNIR) at the array output. Before calibration, the mainlobe lies at 273° and a sharp null is placed at 120°, removing the desired signal from the array output. However, after performing the proposed calibration approach, the peak of the mainlobe lies at 124° and the null in the desired source direction is removed, maximising the SNIR output. Finally, the steering vector beamformer is employed before and after the proposed calibration approach with the setup illustrated in Figure 2 assuming that the source at (azimuth, elevation) = (120°, 0°) is the desired source and that there is no co-channel interference. The array pattern produced in each case is illustrated in Figure 6. Before

![Fig. 2. Nominal, actual and estimated geometry of the array of sensors when the proposed algorithm is employed with the pilot operating from \((\theta, \phi, \rho) = (60°, 0°, 6m)\) at \(F_{c1} = 500\) MHz and \(F_{c2} = 2.45\) GHz under SNR=20dB and \(L = 1000\) snapshots.](image1)

Table 1

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Initial Location Errors</th>
<th>Final Location Errors</th>
</tr>
</thead>
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<td>y-coord (m)</td>
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<td>0</td>
<td>0</td>
</tr>
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<tr>
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</tbody>
</table>

Perfomance of the proposed calibration algorithm under the setup described in Figure 2

![Fig. 3. MUSIC spectrum before and after the proposed calibration approach under the setup described in Figure 2. In both cases there are 3 signals with DOA’s \((\theta, \phi) = (30°, 0°), (35°, 0°)\) and \((120°, 0°)\) observed under SNR=20dB and \(L = 1000\) snapshots.](image2)

![Fig. 4. Array pattern of an interference cancellation beamformer before and after the proposed calibration approach under the setup described in Figure 2. In both cases the DOA of the desired source is \((\theta, \phi) = (30°, 0°)\) and there are two interferences at \((35°, 0°)\) and \((120°, 0°)\).](image3)
calibration, the mainlobe lies at 122° with a gain of 7.54 dB. This indicates the presence of a 2° pointing error implying the performance of the beamformer in the desired source direction will be reduced. However, following calibration, the mainlobe lies at the desired direction of 120° implying the pointing error has been removed and the beamformer has a larger gain of 8.451 dB.

These examples show that the introduction of array location uncertainties have different effects in the performance of the different types of beamformers. In each case, the improvement as a result of the proposed calibration algorithm is clearly illustrated.

V. CONCLUSION

In this paper an array shape calibration algorithm is presented for a planar array which only requires one source transmitting on at least two carrier frequencies. The method exploits the change in signal model as the transmitting source moves from the “near-far” to the “far” field of the array which is achieved by changing the wavelength of the transmitted signal. This provides an effective calibration approach when a sparse number of pilot sources exist within the calibration environment and has been shown to improve the array performance when sensor location uncertainties are present. The method can be extended to perform global calibration (i.e. gain, phase and mutual coupling).

REFERENCES