Decision Rules

1. The receiver of a binary communication system was designed in an optimum way based on the following information:

- $C_{00} = C_{11} = 0; C_{10} = 3; C_{01} = 1$
- Where $C_{ij}$ is the cost associated with choosing hypothesis $H_i$ when in fact $H_j$ is true.
- The likelihood functions are

\[
\text{pdf}_{r/H_0}(r) = \frac{1}{3} \text{rect} \left\{ \frac{r}{3} \right\}
\]

and

\[
\text{pdf}_{r/H_1}(r) = A \{r - 2\}
\]

where $r$ is the observed signal at the output of the channel.

(a) Design an optimum receiver. 15%

(b) Find the forward transition matrix $F$ of this binary channel. 10%

2. Consider a binary communication system in which the channel noise is additive Gaussian of zero mean and variance 1, that is $N(0,1)$. The system employs two correlated signals (with time-cross correlation $\rho = 0.5$), and a correlation receiver which operates on the Bayes-decision criterion with the following costs:

- $C_{00} = C_{11} = 0; C_{10} = 1.858; C_{01} = 0.5$

If the communication system is modelled as follows:

(a) estimate its energy utilization efficiency $EUE$. 30%
(b) What is the False Alarm Probability, $p_{FA}$, and the bit error probability, $p_e$, for the above system?

3. Consider a binary communication system in which the channel noise is additive Gaussian of zero mean and variance 1, that is $N(0,1)$. The system employs two correlated signals with cross-correlation coefficient $\rho$, and a correlation receiver which operates on the Bayes-decision criterion with the following costs:

$$C_{00} = C_{11} = 0; C_{10} = 1.858; C_{01} = 0.5$$

If the communication system has an energy utilisation efficiency $EUE = 5.25 \times 10^{-2}$ and is modelled as follows:

(a) estimate the cross correlation coefficient $\rho$. 30%
(b) What is the False Alarm Probability, $p_{FA}$, and the bit error probability, $p_e$, for the above system? 10%

4. Consider a binary pulse-code-modulation (binary-PCM) system where the digital modulation scheme being used is described as follows:

“The input to the digital modulator is a binary sequence of 1’s and 0’s with the number of 1s being twice the number of zeros. The binary sequence is transmitted as a pulse signal $s(t)$ with a one being sent as $2.\text{rect}\left(\frac{t}{T}\right) + 4\Lambda \left\{ \frac{t}{T_2} \right\}$ and zero being sent as $0.\text{rect}\left(\frac{t}{T}\right)$.”

and the channel noise is assumed to be additive and uniformly distributed between $-2$ Volts and $+2$ Volts

(a) plot the probability density function of $s(t)$ 15%
(b) plot the probability density function of $r(t) = s(t) + n(t)$ 10%
(c) identify the likelihood functions $p_0(r)$ and $p_1(r)$ 15%
(d) design a Bayes Detector (i.e. decision rule) when the following costs apply:

$$C_{00} = C_{11} = 0; C_{10} = 0.8; C_{01} = 1$$

30%

(e) for the above Bayes detector estimate the

- the False Alarm Probability 10%
- the Probability of a Miss 10%
- the bit error probability, $p_e$. 10%

Costellation Diagram

5. The two signals $s_0(t)$ and $s_1(t)$ of a binary communication system each have energy equal to 93.3 and cross correlation coefficient 0.866.

(a) Draw the constellation diagram of the system properly labeled. 10%
(b) What is the distance of these two signals? 10%
6. Consider a binary communication system which uses the following two equiprobable signals

\[ s_0(t) \text{ in Volts:} \]

\[ s_1(t) \text{ in Volts:} \]

These signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of \(10^{-3} \text{ W/Hz}\).

(a) Calculate the values of the associated signal-vectors \( w_s, \forall i \) for the above two signals as a function of the amplitude \( A \). 15%

(b) Draw a labelled block diagram of the MAP correlation receiver based on the signals vectors \( w_s, \forall i \). 15%

(c) Plot the constellation diagram and properly label the decision regions as a function of the amplitude \( A \). 15%

(d) Calculate the amplitude \( A \) needed to achieve a minimum-bit-error probability of \(6 \times 10^{-3}\). 20%

(e) Find the forward transition matrix \( F \) of the equivalent discrete channel. 10%

7. Consider an \( M \)-ary Communication System with its signal set described as follows:

\[ s_i(t) = A_i \left( \frac{2t}{T_{cs}} \right) + \text{rect} \left( \frac{t}{T_{cs}} \right) ; i = 1, 2, ..., M \]  

(1)

\[
\begin{align*}
M &= 4 \\
A_i &= (2i - 1 - M) \times 10^{-3} \text{ Volts} \\
T_{cs} &= 6 \text{ sec} \\
Pr(H_1) &= Pr(H_4) = 0.2 \\
Pr(H_2) &= Pr(H_3) = 0.3
\end{align*}
\]

The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of \(10^{-6} \text{ W/Hz}\).

(a) Find and plot the power spectral density of the transmitted signal \( s(t) \). 10%

(b) Calculate the values of the signal-vectors \( w_s, i = 1, 2, 3, 4 \) for the above signal-set. 20%

(c) Plot the constellation diagram and label the decision regions. 20%

(d) Draw an optimum receiver, based on the signals vectors \( w_s, i = 1, 2, 3, 4 \). 10%

(e) Model the whole system as a discrete communication channel. 15%

(f) Find

- the symbol error probability \( p_{e,cs} \) at the output of the receiver 10%
- the joint-probability matrix \( J \) (i.e. the matrix with elements the probabilities \( Pr(H_i, D_j) \) \( \forall i,j \)) 5%
- the amount of information (bits per channel symbol) delivered at the output of the system/channel. 10%
8. Consider an $M$-ary communication system involving $M = 4$ signals $s_1(t), s_2(t), s_3(t)$ and $s_4(t)$ with energies $E_1, E_2, E_3$ and $E_4$ respectively. If the system uses the following MAP receiver

where $r(t)$ denoted the received signal, $c_1(t)$ and $c_2(t)$ are two orthonormal signals and the matrix $\mathbf{W}$ is defined as follows:

$$
\mathbf{W} = \begin{bmatrix}
\sqrt{3}, & 0, & -\sqrt{3}, & -\sqrt{3} \\
0, & \sqrt{2}, & -2\sqrt{2}, & \sqrt{2}
\end{bmatrix}
$$

(a) Plot the constellation diagram.
(b) Find the cross-correlation coefficients $\rho_{2,4}$ and $\rho_{3,4}$.
(c) Find the probabilities $\Pr(s_1), \Pr(s_2), \Pr(s_3)$ and $\Pr(s_4)$.
(d) Find the energies $E_1, E_2, E_3$ and $E_4$.

9. Given the signalling waveforms $s_1(t), s_2(t), s_3(t)$ and $s_4(t)$ as shown below

find:

(a) The number $D$ of a set of orthonormal signals $\{c_i(t)\}$ that can be used to represent the above waveforms.
(b) The cross-correlation coefficients $\rho_{2,3}$ and $\rho_{3,4}$.
(c) The weight vector $\mathbf{w}_{s_2}$.
(d) The minimum distance of this set of signals.
10. Given the signalling waveforms \( s_1(t), s_2(t), s_3(t) \) and \( s_4(t) \) as shown below:

\[ s_1(t), s_2(t), s_3(t), s_4(t) \]

find:

(a) the minimum number of dimensions required to represent these waveforms in \( N \)-dimensional vector space 10%
(b) an orthonormal set of \( N \) signals \( \{ c_i(t) \} \) that can be used to represent the above waveforms 30%
(c) the values of the associated signal-vectors \( w_{s_i}, \forall i \) 10%
(d) the minimum distance between the signal-vectors \( w_{s_i}, \forall i \) 10%

**Matched Filters**

11. A matched filter is used to detect the signal \( s(t) \)

\[ s(t) = 3 \text{rect} \left\{ \frac{t}{10^{-6}} \right\} \]

which is corrupted by additive white Gaussian noise of double-sided power spectral density \( 0.5 \times 10^{-6} \).

What is the maximum Signal-to-Noise ratio at the filter output? 10%

12. Find the impulse response of an approximate-matched filter matched to the signal \( A \left( \frac{\tau}{T} \right) \) in the presence of non-white noise of autocorrelation function

\[ R_{nn}(\tau) = A \left( \frac{\tau}{T} \right) \]

13. A matched filter is used to detect the signal \( s(t) \)

\[ s(t) = \begin{cases} A, & \text{if } 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \]

which is corrupted by additive white Gaussian noise. What is the peak Signal-to-Noise ratio at the filter output? 10%

14. Consider an \( M \)-ary communication system involving \( M \) equiprobable orthogonal signals \( s_i(t), i = 1, 2, ..., M, 0 < t < T_{cs} \) of equal energy \( E \). The system operates in the presence of additive white Gaussian noise of double sided power spectral density \( N_0/2 \) which is bandlimited to \( B \) Hz.

If \( r(t) \) represents the received signal at the input of a correlation receiver and \( G_j \) is the output of its \( j \)-th correlator (decision variable), defined as

\[ G_j = \int_0^{T_{cs}} r(t).s_j(t).dt \]

find the quantity

\[ \mathcal{E}\{G_j|H_k\} \]

where \( H_k \) denotes the hypothesis that the signal \( s_k(t) \) was sent and \( \mathcal{E}\{.\} \) is the expectation operator. 10%
15. Prove that the maximum signal-to-noise ratio $\text{SNR}_{\text{max}}^{\text{out}}$ at the output of a matched filter is given by:

$$\text{SNR}_{\text{max}}^{\text{out}} = \int_0^T h_o(z).s(T - z).dz$$

where $h_o(t)$ is the impulse response of the filter matched to the signal $s(t)$.

More on MAP Receivers & Constellation Diagram

16. Consider an $M$-ary communication system with its signal set described as follows:

$$s_i(t) = A_i \cos(2\pi F_c t), \quad i = 1, 2, ..., M, 0 < t < 2 \text{ sec}$$

with

$$M = 4$$

$$A_i = (2i - 1 - M) \times 10^{-3} \text{ Volts}$$

$$\Pr(H_1) = \Pr(H_4) = 0.2 \quad \text{and} \quad \Pr(H_2) = \Pr(H_3) = 0.3$$

The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of $10^{-6} \text{ W/Hz}.$

(a) Draw a labelled block diagram of the MAP receiver.  \[5 \text{ marks}\]

(b) Plot the constellation diagram and label the decision regions.  \[5 \text{ marks}\]