

PROBLEM SHEET - FT Tables Refresher

EE3-27: Principles of Classical and Modern Radar

Contents

1	Evaluating Functions of δ -lines	1
2	Auto-Correlation Function and PSD(f)	2
3	Transfer Function and PSD(f)	3
4	Differential Circuit	4
5	Impulse Response and PSD(f)	5
6	PSD(f) at Bandpass Filter's Output	6

1 Evaluating Functions of δ -lines

Evaluate:

$$(a) \int_{-\infty}^{\infty} (t^4 - 3t + 1) \cdot \delta(t - 2) \cdot dt$$
$$\int_{-\infty}^{\infty} (t^4 - 3t + 1) \cdot \delta(t - 2) \cdot dt = (t^4 - 3t + 1)|_{t=2} = 2^4 - 3 \times 2 + 1 = 11$$

$$(b) \int_{-\infty}^{\infty} (\cos(4\pi t) * \delta(t + \frac{1}{4})) \cdot \delta(t - \frac{1}{8}) \cdot dt$$
$$\int_{-\infty}^{\infty} (\cos(4\pi t) * \delta(t + \frac{1}{4})) \cdot \delta(t - \frac{1}{8}) \cdot dt = \int_{-\infty}^{\infty} \cos(4\pi(t + \frac{1}{4})) \cdot \delta(t - \frac{1}{8}) \cdot dt =$$
$$= \cos(4\pi(t + \frac{1}{4}))|_{t=\frac{1}{8}} = \cos(4\pi(\frac{1}{8} + \frac{1}{4})) = \cos(\frac{3}{2}\pi) = 0$$

$$(c) \int_{-\infty}^{\infty} (t^3 - 3t^2 - 11) \cdot \delta(t - 1) \cdot dt \quad (5\%)$$
$$\int_{-\infty}^{\infty} (t^3 - 3t^2 - 11) \cdot \delta(t - 1) \cdot dt = (t^3 - 3t^2 - 11)|_{t=1} = 1^3 - 3 \times 1^2 - 11 = -13$$

$$(d) \int_{-\infty}^{\infty} \{(\sin(4\pi t) * \delta(t + \frac{1}{4}))\} \cdot \delta(t - \frac{1}{4}) \cdot dt \quad (5\%)$$
$$\int_{-\infty}^{\infty} \{(\sin(4\pi t) * \delta(t + \frac{1}{4}))\} \cdot \delta(t - \frac{1}{4}) \cdot dt = \int_{-\infty}^{\infty} (\sin(4\pi(t + \frac{1}{4}))) \cdot \delta(t - \frac{1}{4}) \cdot dt =$$
$$\sin(4\pi(t + \frac{1}{4}))|_{t=\frac{1}{4}} = \sin(4\pi(\frac{1}{4} + \frac{1}{4})) = \sin(2\pi) = 0$$

$$(e) \int_{-\infty}^{\infty} (t^3 - 2t^2 + 1) \cdot \delta(t - 2) \cdot dt \quad (5\%)$$
$$\int_{-\infty}^{\infty} (t^3 - 2t^2 + 1) \cdot \delta(t - 2) \cdot dt = (t^3 - 2t^2 + 1)|_{t=2} = 2^3 - 2 \times 2^2 + 1 = 1$$

$$(f) \int_{-\infty}^{\infty} \{(\cos(2\pi t) * \delta(t - \frac{1}{4}))\} \cdot \delta(t - \frac{1}{12}) \cdot dt \quad (5\%)$$
$$\int_{-\infty}^{\infty} (\cos(2\pi t) * \delta(t - \frac{1}{4})) \cdot \delta(t - \frac{1}{12}) \cdot dt = \int_{-\infty}^{\infty} \cos(2\pi(t - \frac{1}{4})) \cdot \delta(t - \frac{1}{12}) \cdot dt =$$
$$= \cos(2\pi(t - \frac{1}{4}))|_{t=\frac{1}{12}} = \cos(2\pi(\frac{1}{12} - \frac{1}{4})) = \cos(-\frac{1}{3}\pi) = \frac{1}{2}$$

(g) $h(3)$ where $h(t) = (t \cdot \text{rect}\{\frac{t}{8}\}) * \delta(t+3)$ (5%)

$$h(t) = (t \cdot \text{rect}\{\frac{t}{8}\}) * \delta(t+3) = (t+3) \cdot \text{rect}\{\frac{t+3}{8}\}$$

$$\Rightarrow h(3) = 0$$

(h) $h(3)$ where $h(t) = (t \cdot \text{rect}\{\frac{t}{8T}\}) * \delta(t-2)$ (10%)

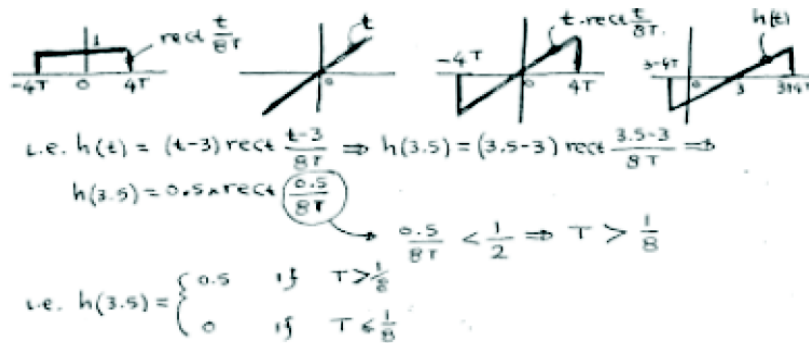
$$h(t) = (t \cdot \text{rect}\{\frac{t}{8T}\}) * \delta(t-2) = (t-2) \cdot \text{rect}\{\frac{t-2}{8T}\}$$

$$h(3) = (3-2) \cdot \text{rect}\{\frac{3-2}{8T}\}$$

$$\Rightarrow h(t) = \begin{cases} t-2 & \text{if } -0.5 < \frac{t-2}{8T} < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

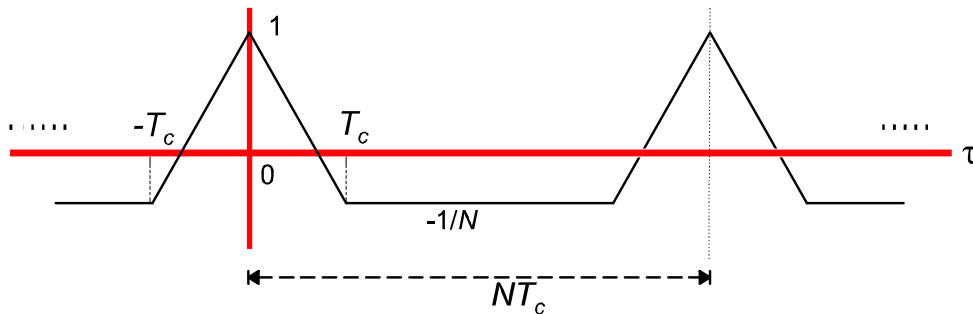
$$\Rightarrow h(3) = \begin{cases} 1 & \text{if } -0.5 < \frac{3-2}{8T} < 0.5 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } T > \frac{1}{4} \\ 0 & \text{if } T \leq \frac{1}{4} \end{cases}$$

(i) $h(3.5)$ where $h(t) = (t \cdot \text{rect}\{\frac{t}{8T}\}) * \delta(t-3)$ (10%)



2 Auto-Correlation Function and PSD(f)

The waveform below shows the autocorrelation function $R_{bb}(\tau)$ of what is called in communications a pseudo-random (PN) signal $b(t)$.



(a) Write a mathematical expression, using Woodward's notation, to describe the above autocorrelation function. (15%)

(b) Find the power spectral density $\text{PSD}_b(f)$ of $b(t)$. (20%)

Solution

(a) $R_{bb}(\tau) = \frac{N+1}{N} \text{rep}_{NT_c} \left\{ \Lambda \left(\frac{\tau}{T_c} \right) \right\} - \frac{1}{N}$

(b) $\text{PSD}(f) = \text{FT}\{R_{bb}(\tau)\} = \frac{N+1}{N^2} \text{comb}_{\frac{1}{NT_c}} \left\{ \text{sinc}^2(fT_c) \right\} - \frac{1}{N} \delta(f)$

3 Transfer Function and PSD(f)

At the input of a filter there is white Gaussian noise of power spectral density $\text{PSD}_{n_i}(f) = \frac{3}{2}10^{-6}$. If the transfer function of the filter is

$$H(f) = \Lambda \left\{ \frac{f}{10^6} \right\} \exp(-j\phi(f))$$

calculate the power of the signal at the output of the filter. (10%)

Solution

$$\text{PSD}_{n_o}(f) = \text{PSD}_{n_i}(f) \cdot |H(f)|^2$$

$$P_{n_o} = \int_{-\infty}^{+\infty} \text{PSD}_{n_i}(f) \cdot |H(f)|^2 \cdot df$$

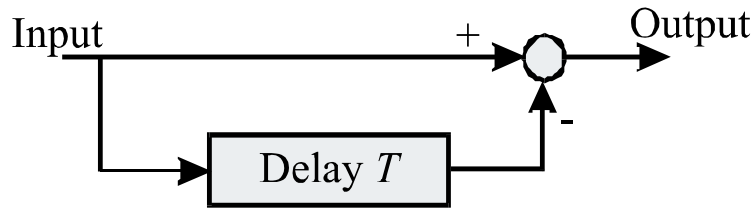
$$= \frac{N_0}{2} \int_{-B}^{+B} |H(f)|^2 df = \frac{N_0}{2} 2 \int_{-B}^{0} |H(f)|^2 df$$

$$= N_0 \int_{-B}^{0} \frac{(f+B)^2}{B^2} df = \frac{N_0}{B^2} \left[\frac{(f+B)^3}{3} \right]_{-B}^{0}$$

$$\Rightarrow P_{n_o} = \frac{N_0 B}{3} = \frac{3 \times 10^{-6} \times 10^6}{3} = 1$$

4 Differential Circuit

For the following differential circuit



find:

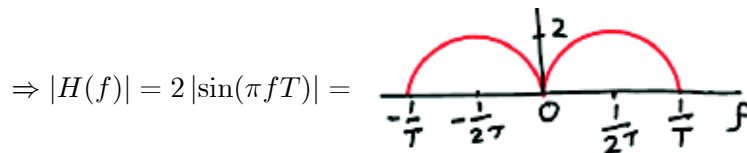
- (a) the impulse response and (5%)
- (b) frequency response (5%)

Solution

(a) $h(t) = \delta(t) - \delta(t - T)$



(b) $H(f) = \text{FT}\{h(t)\} = 1 - \exp(-j2\pi fT)$
 $= \{\exp(j\pi fT) - \exp(-j\pi fT)\} \exp(-j\pi fT)$
 $= 2j \sin(\pi fT) \exp(-j\pi fT)$



5 Impulse Response and PSD(f)

Consider the filter with impulse response

$$h(t) = \text{sinc}^2 \{10^6(t - 3)\}$$

and assume that the input signal $n_i(t)$ is white Gaussian noise with double-sided power spectral density $\text{PSD}_{n_i}(f) = 1.5 \times 10^{-6} \text{ W/Hz}$.

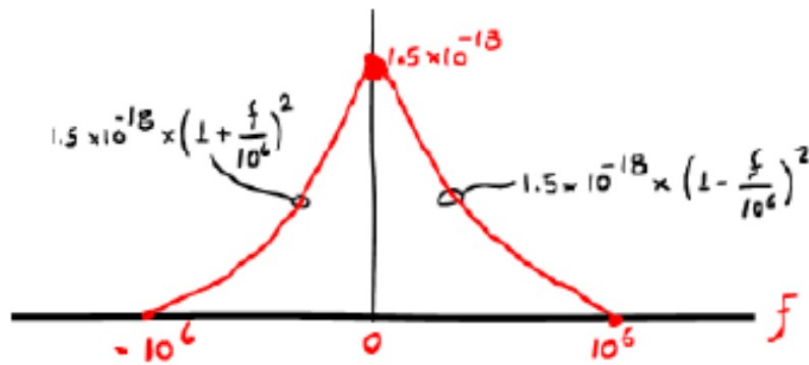
For the signal $n(t)$ at the output of the filter

(a) find and plot its power spectral density $\text{PSD}_n(f)$; (10%)

(b) calculate its power P_n (5%)

Solution

$$\begin{aligned} \text{(a) } H(f) &= \text{FT}\{h(t)\} = \frac{1}{10^6} \Lambda\left(\frac{f}{10^6}\right) \exp(-j2\pi f \times 3) \\ &\Rightarrow \text{PSD}_n(f) = \text{PSD}_{n_i}(f) \cdot |H(f)|^2 = \\ &1.5 \times 10^{-6} \left(\frac{1}{10^6}\right)^2 \Lambda^2\left(\frac{f}{10^6}\right) = 1.5 \times 10^{-18} \Lambda^2\left(\frac{f}{10^6}\right) \end{aligned}$$



$$\begin{aligned} \text{(b) } P_n &= \int_{-10^6}^{10^6} \text{PSD}_n(f) \cdot df = 2 \int_0^{10^6} \text{PSD}_n(f) \cdot df = \\ &= 2 \int_0^{10^6} \left(1.5 \times 10^{-18} \times \left(1 - \frac{f}{10^6}\right)^2\right) \cdot df = \\ &= 2 \times 1.5 \times 10^{-18} \times \int_0^{10^6} \left(1 - \frac{f}{10^6}\right)^2 \cdot df = \\ &= 3 \times 10^{-18} \times \int_0^{10^6} \left(1 - 2\frac{f}{10^6} + \frac{f^2}{10^{12}}\right) \cdot df \\ &= 3 \times 10^{-18} \times \left(10^6 - 2\frac{10^{12}}{2 \times 10^6} + \frac{10^{18}}{3 \times 10^{12}}\right) \\ &= 3 \times 10^{-18} \times \left(10^6 - 10^6 + \frac{1}{3}10^6\right) \\ &= 10^{-12} \end{aligned}$$

6 PSD(f) at Bandpass Filter's Output

Consider a bandpass filter with impulse response

$$h(t) = 8 \times 10^3 \text{sinc}\{4 \times 10^3 t\} \cdot \cos(2\pi 10^4 t)$$

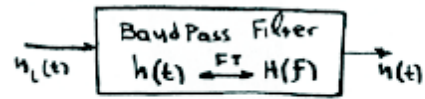
and assume that at the input of this filter there is white Gaussian noise $n_i(t)$ of power spectral density $\text{PSD}_{n_i}(f) = 10^{-6}$.

For the signal $n(t)$ at the output of the filter

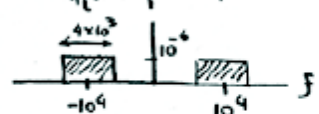
(a) find and plot its power spectral density $\text{PSD}_n(f)$; (10%)

(b) calculate its power P_n (5%)

Solution



$$\begin{aligned}
 H(f) &= \text{FT}\{h(t)\} = 8 \times 10^3 \text{ FT}\{\text{sinc}(4 \times 10^3 t) \cos(2\pi 10^4 t)\} \\
 &= 8 \times 10^3 \text{ FT}\{\text{sinc}(4 \times 10^3 t)\} \otimes \left[\frac{1}{2} \delta(f - 10^4) + \frac{1}{2} \delta(f + 10^4) \right] \\
 &\quad \text{convolution} \\
 &\quad \underbrace{\frac{1}{4 \times 10^3} \text{rect} \frac{f}{4 \times 10^3}} \\
 &= \text{rect} \frac{f - 10^4}{4 \times 10^3} + \text{rect} \frac{f + 10^4}{4 \times 10^3} \\
 \text{PSD}_n(f) &= \text{PSD}_{n_i}(f) \cdot |H(f)|^2 = 10^{-6} \text{rect} \frac{f - 10^4}{4 \times 10^3} + 10^{-6} \text{rect} \frac{f + 10^4}{4 \times 10^3} \\
 &\quad \begin{array}{c} \xrightarrow{4 \times 10^3} \\ \text{rect} \\ \downarrow 10^{-6} \\ \text{---} f \\ \uparrow 10^4 \end{array}
 \end{aligned}$$



$$P_n = 4 \times 10^3 \times 10^{-6} \times 2 = 8 \times 10^{-3} \text{ W} = \underline{\underline{8 \text{ mW}}}$$

END