Topic 6: Decision Theory

v.18

1. The receiver of a binary communication system was designed in an optimum way based on the following information:

   - $C_{00} = C_{11} = 0$; $C_{10} = 3$; $C_{01} = 1$
     where C_{ij} is the cost associated with choosing hypothesis H_i when in fact H_j is true $(i, j = 0, 1)$
   - the likelihood functions are

   $$\mathrm{pdf}_{\mathrm{r}/H_0}(r) = \frac{1}{3}\mathrm{rect}\left\{\frac{r}{3}\right\}$$

   and

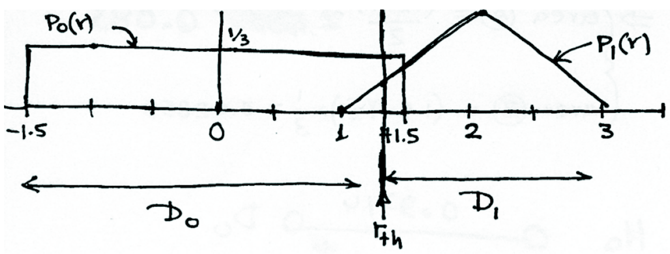
   $$\mathrm{pdf}_{\mathrm{r}/H_1}(r) = \Lambda\left\{r - 2\right\}$$

   where r is the observed signal at the output of the channel.

   (a) Design an optimum receiver.

   (b) Find the forward transition matrix $\mathbb{F}$ of this binary channel.

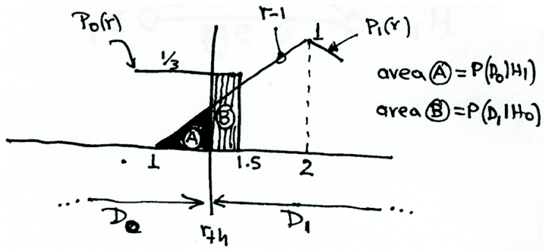
Solution



(a) $C_{00}.\,\Pr(D_0|H_0) + C_{10}.\,\Pr(D_1|H_0) = C_{11}.\,\Pr(D_1|H_1) + C_{01}.\,\Pr(D_0|H_1)$
$\Rightarrow 3\,\Pr(D_1|H_0) = \Pr(D_0|H_1)$ Equ(1)

$\text{Equ}(1) \Rightarrow \underbrace{3(\mathbf{1.5}-r_{\text{th}})\,\frac{\mathbf{1}}{\mathbf{3}}}_{\text{area B}} = \underbrace{\frac{(r_{\text{th}}-\mathbf{1})^{\mathbf{2}}}{\mathbf{2}}}_{\text{area A}}$

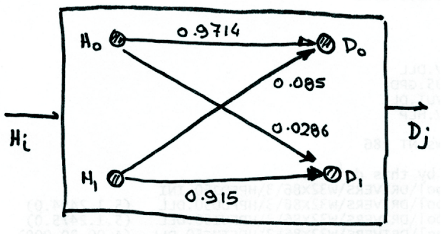
$\Rightarrow (1.5-r_{\text{th}}) = \frac{r_{\text{th}}^2 - 2r_{\text{th}} + 1}{2}$
$\Rightarrow 3 - 2r_{\text{th}} = r_{\text{th}}^2 - 2r_{\text{th}} + 1$
$\Rightarrow r_{\text{th}}^2 = 2$
$\Rightarrow r_{\text{th}} = \sqrt{2} = 1.41$



(b) $\mathbb{F}$=?

$$r_{\text{th}} = \sqrt{2} = 1.41 \Rightarrow \begin{cases} \text{area A} = \frac{(\sqrt{2}-1)^2}{2} \simeq \mathbf{0.085} \\ \text{area B} = (1.5-\sqrt{2})\frac{1}{3} \simeq \mathbf{0.02859} \end{cases}$$

$$\text{i.e. } \mathbb{F} = \begin{bmatrix} 0.9714, & 0.085 \\ 0.0286, & 0.915 \end{bmatrix}$$



Minimax : $\text{weight}_0 \cdot \text{pdf}_{r|H_0} = \text{weight}_1 \cdot$

2. Consider a binary pulse-code-modulation (binary-PCM) system where the digital modulation scheme being used is described as follows:
"The input to the digital modulator is a binary sequence of 1's and 0's with the number of 1s being twice the number of zeros. The binary sequence is transmitted as a pulse signal $s(t)$ with a *one* being sent as $2.\mathrm{rect}\left\{\frac{t}{T_b}\right\}$ $+4\Lambda\left\{\frac{t}{T_b/2}\right\}$ and *zero* being sent as $0.\mathrm{rect}(\frac{t}{T_b})$."
and the channel noise is assumed to be additive and uniformly distributed between -2 Volts and $+2$ Volts

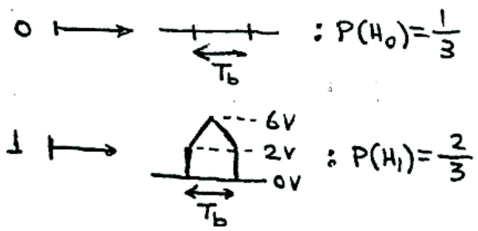
   (a) plot the probability density function of $s(t)$
   (b) plot the probability density function of $r(t) = s(t) + n(t)$
   (c) identify the likelihood functions $p_0(r)$ and $p_1(r)$
   (d) design a Bayes Detector (i.e. decision rule) when the following costs apply:

$$C_{00} = C_{11} = 0; C_{10} = 0.8; C_{01} = 1$$

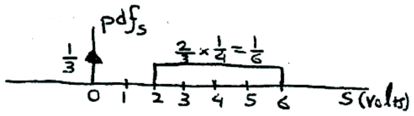
   (e) for the above Bayes detector estimate the
      - the False Alarm Probability
      - the Probability of a Miss
      - the bit error probability, p_e.

Solution

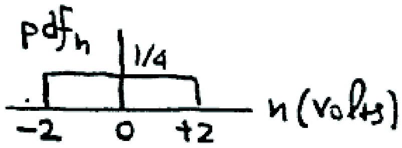
(a) pdf of $s(t)$:



$\Rightarrow \mathrm{pdf}_s(s) = \frac{1}{3}\delta(s) + \frac{2}{3} \times \frac{1}{4} \times \mathrm{rect}\left\{\frac{s-4}{4}\right\}$
i.e.

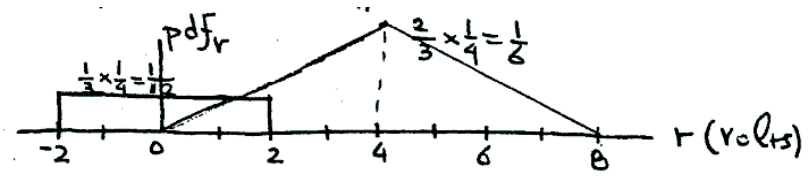


(b) pdf_n :

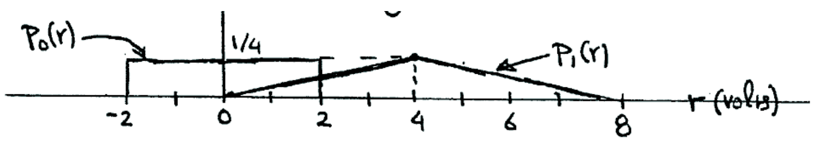


$$\begin{aligned} r(t) &= s(t)+n(t) \\ \Rightarrow \mathrm{pdf}_r &= \mathrm{pdf}_s * \mathrm{pdf}_n \\ &= \underbrace{\frac{1}{3}}_{\Pr(H_0)}\times \underbrace{\frac{1}{4}\times \mathrm{rect}\left\{\frac{r}{4}\right\}}_{\mathrm{pdf}_{r|H_0}} + \underbrace{\frac{2}{3}}_{\Pr(H_1)}\times \underbrace{\frac{1}{4}\times 4\times \Lambda\left\{\frac{r-4}{4}\right\}}_{\mathrm{pdf}_{r|H_1}} \end{aligned}$$

i.e.i.e. $\mathrm{pdf}_r(r) = \frac{1}{12}\mathrm{rect}\frac{r}{4} + \frac{2}{12}\Lambda\left\{\frac{r-4}{4}\right\}$

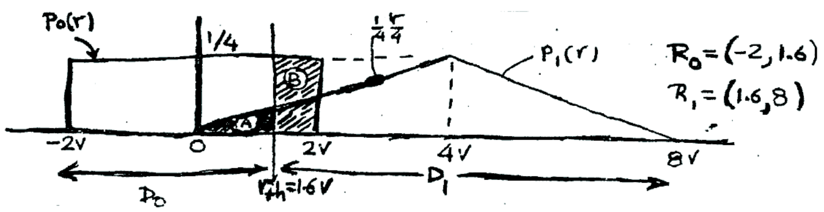


(c) likelihood functions placed together on the same graph:i.e.



$p_0(r) \triangleq \mathrm{pdf}_{r/H_0}(r) = \frac{1}{4}\ \mathrm{rect}\left\{\frac{r}{4}\right\}$, $\quad p_1(r) \triangleq \mathrm{pdf}_{r/H_1}(r) = \frac{1}{4}\ \Lambda\left\{\frac{r-4}{4}\right\}$

(d) Bayes detector:
Choose H_1 iff
$(C_{10}-C_{00})\Pr(H_0)\mathrm{pdf}_{r/H_0}(r) < (C_{01}-C_{11})\Pr(H_1)\mathrm{pdf}_{r/H_1}(r)$
$\Rightarrow 0.8\times\frac{1}{3}\times\frac{1}{4}\ \mathrm{rect}\left\{\frac{r}{4}\right\} < 1\times\frac{2}{3}\times\frac{1}{4}\ \Lambda\left\{\frac{r-4}{4}\right\}$
$\Rightarrow 0.4\ \mathrm{rect}\left\{\frac{r}{4}\right\} < \Lambda\left\{\frac{r-4}{4}\right\} \Rightarrow 0.4 < \frac{r}{4} \Rightarrow r > 1.6$Volts
i.e. $\boxed{\text{choose } H_1 \text{ iff } r > 1.6\text{V, otherwise choose } H_0}$



(e) $P_{FA} = \Pr(D_1|H_0) =$area $\boxed{\mathrm{B}} = \frac{1}{4}\times 0.4 = 0.1$
$P_{miss} = \Pr(D_0|H_1) =$area $\boxed{\mathrm{A}} = \int_0^{1.6} \frac{1}{4}\frac{r}{4}dr = \frac{1}{16}\frac{r^2}{2}\Big]_0^{1.6} = \frac{1}{32}\times 2.56 = 0.08$

- $\mathbb{F} = \begin{bmatrix} 0.9, & 0.08 \\ 0.1, & 0.92 \end{bmatrix}, \underline{p} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$
- $p_e = ?$

$$\begin{aligned} p_e &= \Pr(D_1|H_0).\Pr(H_0) + \Pr(D_0|H_1).\Pr(H_1) \\ &= 0.1\times\frac{1}{3} + 0.08\times\frac{2}{3} \\ &= 0.0867 \end{aligned}$$

3. Consider a binary pulse-code-modulation (binary-PCM) system where the digital modulation scheme being used is described as follows:

   "The input to the digital modulator is a binary sequence of 1's and 0's with the number of 1s being twice the number of zeros. The binary sequence is transmitted as a pulse signal $s(t)$ with a *one* being sent as $\text{rect}(\frac{t}{T_b}) + 4\Lambda\left\{\frac{t}{T_b/2}\right\}$ and *zero* being sent as $2\text{rect}(\frac{t}{T_b}) - 4\Lambda\left\{\frac{t}{T_b/2}\right\}$."

   and the channel noise is assumed to be additive and uniformly distributed between -2 Volts and $+2$ Volts. Find the two likelihood functions of the above system.

   or

   Consider a binary pulse-code-modulation (binary-PCM) system where the input to the digital modulator is a binary sequence of 1's and 0's with the number of 1's being twice the number of zeros.

   The binary sequence is transmitted as a pulse signal $s(t)$ with a *one* being sent as $\text{rect}(\frac{t}{T_b}) + 4\Lambda\left\{\frac{t}{T_b/2}\right\}$ and *zero* being sent as $2\text{rect}(\frac{t}{T_b}) - 4\Lambda\left\{\frac{t}{T_b/2}\right\}$.

   The channel noise is assumed to be additive and uniformly distributed between -2 Volts and $+2$ Volts. Find:

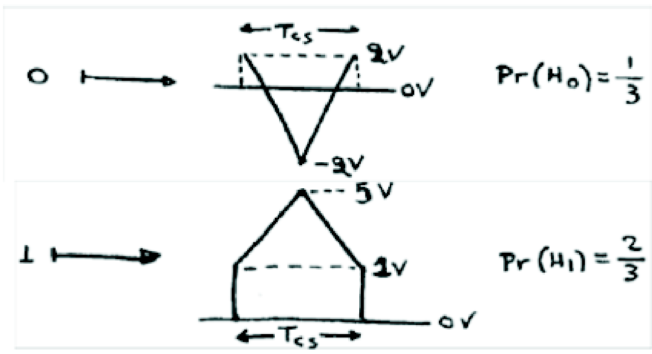
   (a) the probability density function $\text{pdf}_s(s)$, of the transmitted signal $s(t)$;

   (b) the probability density function, $\text{pdf}_r(r)$, of the received signal $r(t)$;

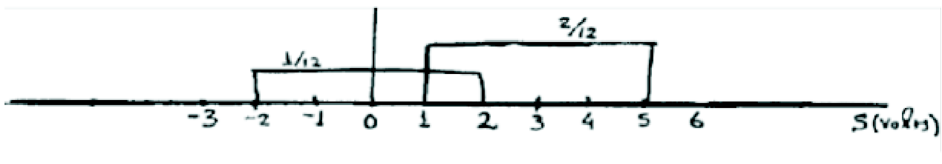
   (c) the likelihood functions of the above system.

   <u>Answer</u>

   (a) Probability density function $\text{pdf}_s(s)$

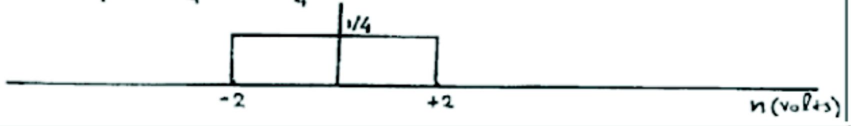


$$\text{pdf}_s(s) = \frac{1}{3} \times \frac{1}{4}\,\text{rect}\,\frac{s}{4} + \frac{2}{3} \times \frac{1}{4}\,\text{rect}\,\frac{s-3}{4}$$



(b) Probability density function, $\mathrm{pdf}_r(r)$

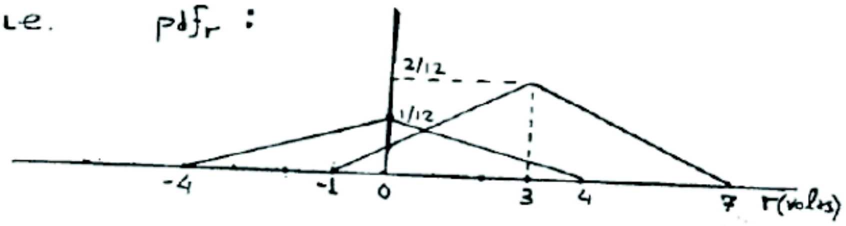
$$\mathrm{pdf}_n(n) = \frac{1}{4}\,\mathrm{rect}\,\frac{n}{4}$$



$$r(t) = s(t) + n(t) \;\Rightarrow\; \mathrm{pdf}_r(r) = \mathrm{pdf}_s \underset{\substack{\uparrow\\ \text{convolution}}}{*} \mathrm{pdf}_n$$

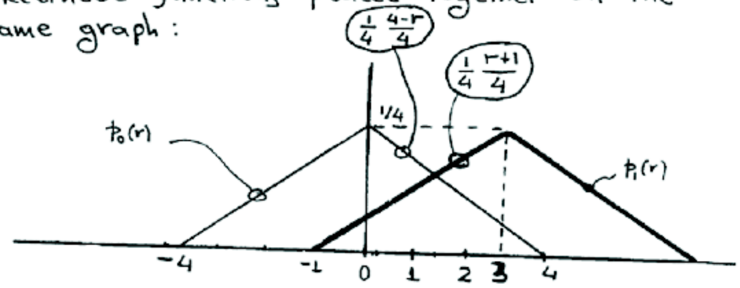
$$\Rightarrow \mathrm{pdf}_r(r) = \frac{1}{3} \times \underbrace{\frac{1}{4} \times \frac{4}{4}\,\Lambda\left\{\frac{r}{4}\right\}}_{\mathrm{pdf}_{r|H_0} \equiv p_0(r)} + \frac{2}{3} \times \underbrace{\frac{1}{4} \times \frac{4}{4}\,\Lambda\left\{\frac{r-3}{4}\right\}}_{\mathrm{pdf}_{r|H_1} \equiv p_1(r)}$$

i.e. pdf_r :



(c) Likelihood Functions

likelihood functions placed together on the same graph :



$$\text{i.e.}\quad \left.\begin{aligned} p_0(r) &= \frac{1}{4}\,\Lambda\left\{\frac{r}{4}\right\} \\ p_1(r) &= \frac{1}{4}\,\Lambda\left\{\frac{r-3}{4}\right\} \end{aligned}\right\} \Rightarrow \text{likelihood ratio } \ell(r) = \frac{\Lambda\left\{\frac{r-3}{4}\right\}}{\Lambda\left\{\frac{r}{4}\right\}}$$

4. Consider an M-ary Communication System for which the signal set is described as follows:

$$s_i(t) = A_i \text{rect}\left\{\frac{t}{T_{cs}}\right\},\ i = 1, 2, ..., M.$$

$$\text{with} \begin{cases} M = 4 \\ A_i = (2i - 1 - M) \times 10^{-3}\text{Volts} \\ T_{cs} = 4\,\text{sec} \\ \Pr(\text{H}_1) = \Pr(\text{H}_4) = 1/8 \ \text{ and } \ \Pr(\text{H}_2) = \Pr(\text{H}_3) = 3/8 \end{cases}$$

The signals are transmitted over a communication channel which adds white Gaussian noise with a double-sided power spectral density of 10^{-6} W/Hz.

(a) Find the energy $E_i, i = 1, 2, 3, 4$.

(b) Calculate the values of the signal-vectors $\underline{w}_{s_i}$, $i = 1, 2, 3, 4$ for the above signal-set.

(c) Draw a labelled block diagram of the MAP receiver, based on the signals vectors $\underline{w}_{s_i}, i =$ 1,2,3,4.

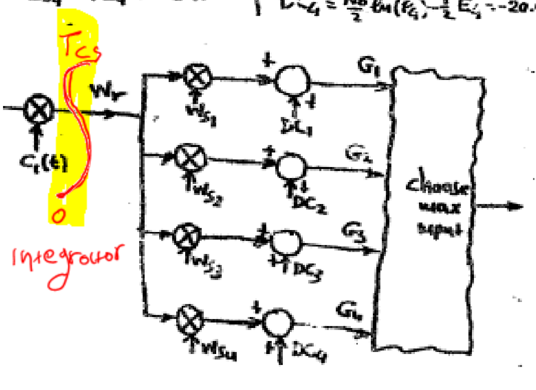
(d) Plot the constellation diagram and label the decision regions.

Answer

$$E_i = \int_{T_{cs}/2}^{T_{cs}/2} A_i^2 \text{rect}\left\{\tfrac{t}{T_{cs}}\right\} dt = A_i^2 T_{cs} = 4A_i^2 = 36 \times 10^{-6}, 4 \times 10^{-6}, 4 \times 10^{-6}, 36 \times 10^{-6}$$

$$c_1(t) = -\tfrac{A_1}{\sqrt{E_1}} \text{rect}\left\{\tfrac{t}{4}\right\} = -\tfrac{1}{2} \text{rect}\left\{\tfrac{t}{4}\right\}$$

$$\left.\begin{aligned} \underline{w}_{s_1} &= -\sqrt{E_1} = -6\text{mV} \\ \underline{w}_{s_2} &= -\sqrt{E_2} = -2\text{mV} \\ \underline{w}_{s_3} &= \sqrt{E_3} = 2\text{mV} \\ \underline{w}_{s_4} &= \sqrt{E_4} = 6\text{mV} \end{aligned}\right\} \quad \begin{aligned} DC_1 &= \tfrac{N_0}{2}\ln(P_1) - \tfrac{1}{2}E_1 = -20.073 \times 10^{-6} \\ DC_2 &= \tfrac{N_0}{2}\ln(P_2) - \tfrac{1}{2}E_2 = -2.98 \times 10^{-6} \\ DC_3 &= \tfrac{N_0}{2}\ln(P_3) - \tfrac{1}{2}E_3 = -2.98 \times 10^{-6} \\ DC_4 &= \tfrac{N_0}{2}\ln(P_4) - \tfrac{1}{2}E_4 = -20.073 \times 10^{-6} \end{aligned}$$

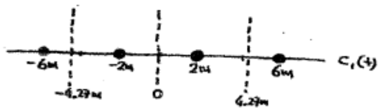


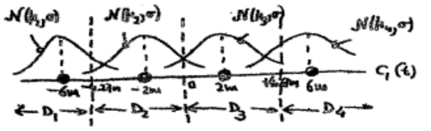
$$G_i = w_r w_{s_i} + DC_i$$

$$G_1 = G_2 \Rightarrow w_r w_{s_1} + DC_1 = w_r w_{s_2} + DC_2 \Rightarrow w_{r_1} = \frac{DC_2 - DC_1}{w_{s_1} - w_{s_2}} = -4.27$$

$$G_2 = G_3 \Rightarrow \cdots \Rightarrow w_{r_2} = \frac{DC_3 - DC_2}{w_{s_2} - w_{s_3}} = 0$$

$$G_3 = G_4 \Rightarrow \cdots \Rightarrow w_{r_3} = \frac{DC_4 - DC_3}{w_{s_3} - w_{s_4}} = +4.27$$





$\mu_1 = -6m$

$\mu_2 = -2m$

$\mu_3 = 2m$

$\mu_4 = 6m$

$$\sigma^2 = \frac{N_0}{2} \underbrace{2B}_{\frac{1}{2T_{cs}}} T_{cs} = \frac{N_0}{2} = 10^{-6} \Rightarrow \boxed{\sigma = 10^{-3}}$$

5. Consider an M-ary Communication System with its signal set described as follows:

$$s_i(t) = A_i \mathbf{\Lambda}\left\{\frac{2t}{T_{cs}}\right\},\ i = 1, 2,\ ...,\ M.$$

$$with \begin{cases} M = 4 \\ A_i = (2i - 1 - M) \times 10^{-3}\text{Volts} \\ T_{cs} = 12\,\text{sec} \\ \Pr(\text{H}_1) = \Pr(\text{H}_4) = 1/8 \ \text{ and } \ \Pr(\text{H}_2) = \Pr(\text{H}_3) = 3/8 \end{cases}$$

The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-6} W/Hz.

(a) Find the power spectral density of the transmitted signal $s(t)$.

(b) Find the energy $E_i, i = 1, 2, 3, 4$.

(c) Calculate the values of the signal-vectors w_{s_i}, $i = 1, 2, 3, 4$ for the above signal-set.

(d) Draw a labelled block diagram of the MAP receiver, based on the signals vectors $w_{s_i}, i =$ 1,2,3,4.

Answer

$$\text{PSD}_s(f) = \frac{1}{T_{cs}}\, \mathcal{E}\left\{\left|FT\left(A_i \Lambda\left(\frac{2t}{T_{cs}}\right)\right)\right|^2\right\}$$

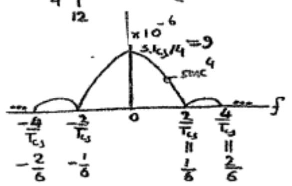
$$= \frac{1}{T_{cs}}\, \mathcal{E}\left\{\left|A_i \frac{T_{cs}}{2} \operatorname{sinc}^2\left(f\frac{T_{cs}}{2}\right)\right|^2\right\}$$

$$= \frac{1}{T_{cs}}\, \mathcal{E}\left\{A_i^2 \frac{T_{cs}^2}{4} \operatorname{sinc}^4\left(f\frac{T_{cs}}{2}\right)\right\}$$

$$= \frac{1}{T_{cs}} \frac{T_{cs}^2}{4} \operatorname{sinc}^4\left(f\frac{T_{cs}}{2}\right) \cdot \mathcal{E}\{A_i^2\}$$

$$= \frac{T_{cs}}{4} \cdot \operatorname{sinc}^4\left(f\frac{T_{cs}}{2}\right)\left((-3)^2\frac{1}{8} + (-1)^2\frac{3}{8} + 1^2\frac{3}{8} + 3^2\frac{1}{8}\right) \times 10^{-6}$$

$$= \frac{3}{4} \underset{\underset{12}{\uparrow}}{T_{cs}} \operatorname{sinc}^4\left(f\frac{T_{cs}}{2}\right) \times 10^{-6} = 9 \operatorname{sinc}^4\left(f\frac{T_{cs}}{2}\right) \times 10^{-6}$$



$$E_i = \int_{-T_{cs}/2}^{T_{cs}/2} A_i^2 \Lambda^2\left(\frac{2t}{T_{cs}}\right) dt =$$

$$= 2\int_{-T_{cs}/2}^{0} A_i^2 \left(\frac{t + T_{cs}/2}{T_{cs}/2}\right)^2 dt =$$

$$= 2A_i^2 \int_{-T_{cs}/2}^{0} \frac{t^2 + T_{cs}^2/4 + 2t\,T_{cs}/2}{T_{cs}^2/4} dt$$

$$= \frac{8A_i^2}{T_{cs}^2} \int_{-T_{cs}/2}^{0} \left(t^2 + 2\frac{T_{cs}}{2}t + \frac{T_{cs}^2}{4}\right) dt$$

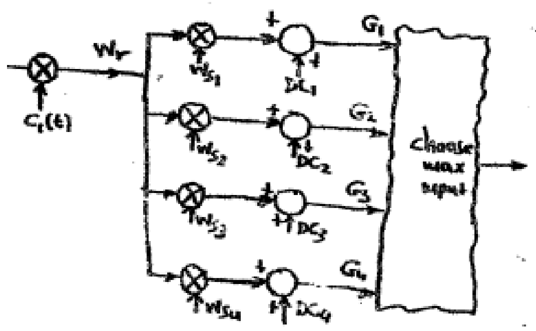
$$= \frac{8A_i^2}{T_{cs}^2}\left(\frac{t^3}{3}\Big|_{-T_{cs}/2}^{0} + \cancel{2}\frac{T_{cs}}{\cancel{2}}\frac{t^2}{2}\Big|_{-T_{cs}/2}^{0} + \frac{T_{cs}^2}{4}t\Big|_{-T_{cs}/2}^{0}\right)$$

$$= \frac{8A_i^2}{T_{cs}^2}\left[\frac{T_{cs}^3}{3\times 8} - \cancel{\frac{T_{cs}^3}{8}} + \cancel{\frac{T_{cs}^3}{8}}\right] = \frac{1}{3} T_{cs} A_i^2 = 4A_i^2$$

$$c_1(t) = -A_1\,\Lambda\left(\frac{2t}{T_{cs}}\right)\Big/\sqrt{E_1} = \sqrt{\frac{3}{T_{cs}}}\,\Lambda\left(\frac{2t}{T_{cs}}\right)$$

ie $\boxed{c_1(t) = \frac{1}{2}\,\Lambda\left(\frac{t}{6}\right)}$

$$\begin{aligned}
\underline{w}_{s_1} &= -\sqrt{E_1} = -6\text{mV} \qquad & DC_1 &= \frac{N_0}{2}\ln(P_1) - \frac{1}{2}E_1 = -20.079\times10^{-6}\\
\underline{w}_{s_2} &= -\sqrt{E_2} = -2\text{mV} \qquad & DC_2 &= \frac{N_0}{2}\ln(P_2) - \frac{1}{2}E_2 = -2.98\times10^{-6}\\
\underline{w}_{s_3} &= \sqrt{E_3} = 2\text{mV} \qquad & DC_3 &= \frac{N_0}{2}\ln(P_3) - \frac{1}{2}E_3 = -2.98\times10^{-6}\\
\underline{w}_{s_4} &= \sqrt{E_4} = 6\text{mV} \qquad & DC_4 &= \frac{N_0}{2}\ln(P_4) - \frac{1}{2}E_4 = -20.079\times10^{-6}
\end{aligned}$$

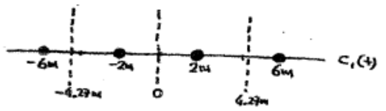


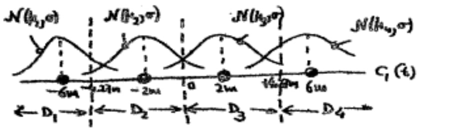
$G_i = w_r w_{s_i} + DC_i$

$G_1 = G_2 \Rightarrow w_r w_{s_1} + DC_1 = w_r w_{s_2} + DC_2 \Rightarrow w_{r_1} = \dfrac{DC_2 - DC_1}{w_{s_1} - w_{s_2}} = -4.27$

$G_2 = G_3 \Rightarrow \cdots \Rightarrow w_{r_2} = \dfrac{DC_3 - DC_2}{w_{s_2} - w_{s_3}} = 0$

$G_3 = G_4 \Rightarrow \cdots \Rightarrow w_{r_3} = \dfrac{DC_4 - DC_3}{w_{s_3} - w_{s_4}} = +4.27$





$\mu_1 = -6\text{m}$
$\mu_2 = -2\text{m}$
$\mu_3 = 2\text{m}$
$\mu_4 = 6\text{m}$

$$\sigma^2 = \frac{N_0}{2}\,2\underbrace{B}_{\frac{1}{2T_{cs}}}\,T_{cs} = \frac{N_0}{2} = 10^{-6} \Rightarrow \boxed{\sigma = 10^{-3}}$$

6. Consider an M-ary Communication System with its signal set described as follows: $s_i(t) = A_i\mathbf{cos}(2\pi F_c t)$, $i = 1, 2, .., M$, $0 < t < 2\,\text{sec}$

$$\text{with} \begin{cases} M = 4 \\ A_i = (2i - 1 - M) \times 10^{-3}\text{Volts} \\ \Pr(\mathrm{H}_1) = \Pr(\mathrm{H}_4) = 0.2 \text{ and } \Pr(\mathrm{H}_2) = \Pr(\mathrm{H}_3) = 0.3 \end{cases}$$

The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-6} W/Hz.

(a) Draw a labelled block diagram of the MAP receiver.

(b) Plot the constellation diagram and label the decision regions.

Solution

(a) MAP Rx:

$N_o = 2\times10^{-6}$

$T_{cs} = 2$

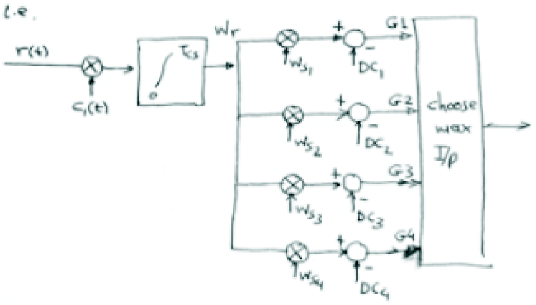
$D = 1$ (dim) $\longrightarrow c_1(t) = \frac{2}{T_{cs}}\cos(2\pi F_c t) = \cos(2\pi F_c t)$

$A_i = (2i-1-4)\times10^{-3} \Rightarrow \begin{cases} A_1 = -3\text{mV} \\ A_2 = -1\text{mV} \\ A_3 = 1\text{mV} \\ A_4 = 3\text{mV} \end{cases}$

$w_{s_i} = -\sqrt{E_i} = -\sqrt{\frac{A_i^2}{2}T_{cs}} \Rightarrow \begin{cases} w_{s_1} = -3\times10^{-3} \\ w_{s_2} = -10^{-3} \\ w_{s_3} = +10^{-3} \\ w_{s_4} = 3\times10^{-3} \end{cases}$

$DC_i = \frac{N_o}{2}\ln(P_i) - \frac{1}{2}E_i \Rightarrow \begin{cases} DC_1 = -6.109\times10^{-6} \\ DC_2 = -1.704\times10^{-6} \\ DC_3 = -1.704\times10^{-6} \\ DC_4 = -6.109\times10^{-6} \end{cases}$

i.e.



Note: $G_i = w_r.w_{s_i} + DC_i$

(b) 1st threshold, w_{r,thr_1} :

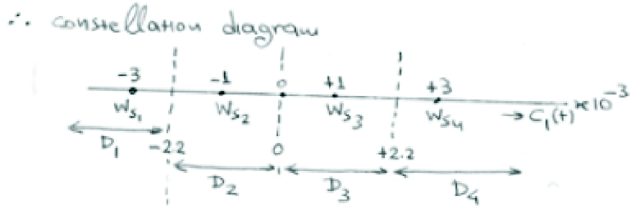
$$\begin{aligned} G_1 &= G_2 \Rightarrow \\ w_{r,thr_1}.w_{s_1} + DC_1 &= w_{r,thr_1}.w_{s_2} + DC_2 \\ w_{r,thr_1} &= \frac{DC_2 - DC_1}{w_{s_1} - w_{s_2}} = -2.2 \times 10^{-3} \end{aligned}$$

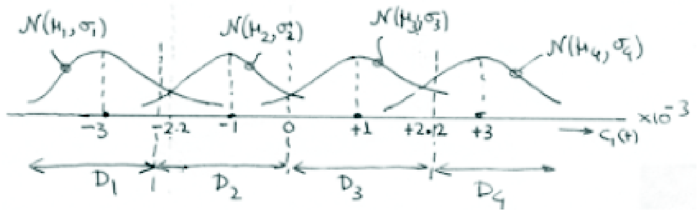
similarly, 2nd threshold w_{r,thr_2} :

$$
\begin{aligned}
G_2 &= G_3 \Rightarrow \\
w_{r,thr_2} &= \frac{DC_3 - DC_2}{w_{s_2} - w_{s_3}} = 0
\end{aligned}
$$

and 3rd threshold w_{r,thr_3} :

$$
\begin{aligned}
G_3 &= G_4 \Rightarrow \\
w_{r,thr_3} &= \frac{DC_4 - DC_3}{w_{s_3} - w_{s_4}} = 2.2 \times 10^{-3}
\end{aligned}
$$





$\mu_1 = -3 \times 10^{-3}$
$\mu_2 = -1 \times 10^{-3}$
$\mu_3 = +1 \times 10^{-3}$
$\mu_4 = +3 \times 10^{-3}$
$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma$
where $\sigma^2 = \frac{N_0}{2} 2 \underset{\underset{2T_{cs}}{\uparrow}}{B} T_{cs} = \frac{N_0}{2} = 10^{-6}$ (this is the noise energy over T_{cs}) $\Rightarrow \sigma = 10^{-3}$

7. Consider a binary communication system which uses the following two equiprobable signals $s_0(t)$ and $s_1(t)$. The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-6} W/Hz. The constellation diagram is given below, where the decision regions are also shown.

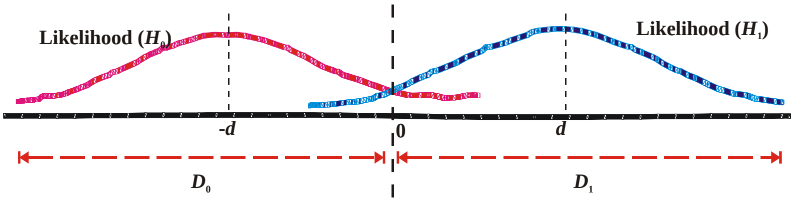


Figure Q10

If the forward transition matrix $\mathbb{F}$ of the equivalent discrete channel is

$$\mathbb{F} = \begin{bmatrix} 0.994, & 0.006 \\ 0.006, & 0.994 \end{bmatrix}$$

calculate the value of the parameter d.

Answer:

$N_0 = 2 \times 10^{-6}$; likelihood functions $\text{pdf}_{r/H_0} = \text{N}(-d, \frac{N_0}{2})$ & $\text{pdf}_{r/H_1} = \text{N}(d, \frac{N_0}{2})$

$\mathbb{F} = \begin{bmatrix} 0.994, & 0.006 \\ 0.006, & 0.994 \end{bmatrix} = \begin{bmatrix} \Pr(D_0|H_0), & \Pr(D_0|H_1) \\ \Pr(D_1|H_0), & \Pr(D_1|H_1) \end{bmatrix}$

$\Rightarrow \Pr(D_1|H_0) = \Pr(D_0|H_1) = \text{T}\left(\frac{d}{\sqrt{N_0/2}}\right) = 0.006$

and $\Pr(D_0|H_0) = \Pr(D_1|H_1) = 1 - \text{T}\left(\frac{d}{\sqrt{N_0/2}}\right) = 0.994$

$p_e = 0.006 = \Pr(D_1|H_0)\Pr(H_0) + \Pr(D_0|H_1)\Pr(H_1) =$

$= \Pr(D_1|H_0)\frac{1}{2} + \Pr(D_0|H_1)\frac{1}{2} = \text{T}\left(\frac{d}{\sqrt{N_0/2}}\right) \Rightarrow 2.5 = \frac{d}{\sqrt{N_0/2}} \Rightarrow$

$d = 2.5 \times \sqrt{N_0/2} = 2.5 \times 10^{-3}$

8. Consider a binary communication system which uses the following two equiprobable signals $s_0(t)$ and $s_1(t)$ of equal energy E and cross correlation $\rho_{01} = -1$. The signals are transmitted over a communication channel which adds white Gaussian noise having a double-sided power spectral density of 10^{-6} W/Hz. If the forward transition matrix $\mathbb{F}$ of the equivalent discrete channel is $\mathbb{F}=\begin{bmatrix} 0.994, & 0.006 \\ 0.006, & 0.994 \end{bmatrix}$ calculate the energy E.

   Answer

   $N_0 = 2 \times 10^{-6}$;

   likelihood functions: $\text{pdf}_{r/H_0} = \text{N}(-d, \frac{N_0}{2})$ & $\text{pdf}_{r/H_1} = \text{N}(d, \frac{N_0}{2})$

   $$\mathbb{F}=\begin{bmatrix} 0.994, & 0.006 \\ 0.006, & 0.994 \end{bmatrix}=\begin{bmatrix} \Pr(D_0|H_0), & \Pr(D_0|H_1) \\ \Pr(D_1|H_0) & \Pr(D_1|H_1) \end{bmatrix}$$

   $$\Rightarrow \Pr(D_1|H_0) = \Pr(D_0|H_1) = \text{T}\left(\frac{d}{\sqrt{N_0/2}}\right) = 0.006$$

   and $\Pr(D_0|H_0) = \Pr(D_1|H_1) = 1 - \text{T}\left(\frac{d}{\sqrt{N_0/2}}\right) = 0.994$

   $$p_e = 0.006 = \Pr(D_1|H_0)\Pr(H_0) + \Pr(D_0|H_1)\Pr(H_1) =$$

   $$= \Pr(D_1|H_0)\tfrac{1}{2} + \Pr(D_0|H_1)\tfrac{1}{2} = \text{T}\left(\frac{d}{\sqrt{N_0/2}}\right) \Rightarrow 2.5 = \frac{d}{\sqrt{N_0/2}} \Rightarrow$$

   $d = 2.5 \times \sqrt{N_0/2} = 2.5 \times 10^{-3}$

   $E = d^2 = 6.25 \times 10^{-6}$