EXTENDED ARRAY MANIFOLDS

by

GEORGIOS EFSATHOPOULOS

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Department of Electrical & Electronic Engineering
Imperial College London
University of London
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Abstract

This thesis is concerned with the investigation of the extended array manifolds and their applications. Extended array manifolds are an extension of the spatial array manifold by incorporating additional channel parameters, such as the CDMA code, the lack of synchronisation, etc. Initially, the geometric properties of these manifolds are derived using differential geometry. Then the theoretical knowledge acquired by this investigation is used to estimate theoretical performance bounds for array systems, analyse the performance of channel estimation algorithms and finally design a novel algorithm for array calibration in a non-stationary signal environment.

Firstly, the concept of the “extended” array manifolds is introduced. This generic model is shown to accommodate both existing and newly defined extensions of the widely employed in the literature spatial array manifold. The geometry of the extended array manifolds is studied and the theoretical results are then readily applied to estimate the geometric properties of various array manifolds.

Furthermore, existing theoretical performance bounds for linear array systems are extended for array systems of arbitrary 3-dimensional geometry. The theoretical tools developed during the analysis of the extended array manifolds are used in order to compare the performance of various array systems employing antenna arrays of identical geometry, but operating in diverse signal environments.

A special class of array manifolds, “hyperhelical” manifolds, is studied next. Their study is motivated by their unique properties, which greatly facilitate the
computability of their geometric parameters. The issues of existence and uniqueness of hyperhelical manifolds is addressed and the various antenna geometries giving rise to hyperhelical array manifolds are identified.

Finally, the problem of array uncertainties is tackled using a geometric approach. The effects of geometric and carrier uncertainties on channel estimation and subspace based, algorithms are investigated and modelled. Based on this modelling, an array calibration algorithm for a non-stationary signal environment is proposed.
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Publications related to this thesis

The following publications contain research work included in this thesis.


Notation

\( a, A \) scalar

\( \mathbf{a}, \mathbf{A} \) column vector

\( \mathbf{a}, \mathbf{A} \) matrix

\( \mathbf{0}_N \) \( N \) element column vector of zeros

\( \mathbf{1}_N \) \( N \) element column vector of ones

\( \mathbb{I}_N \) Identity matrix of size \( N \times N \)

\( \mathbf{0}_{M \times N} \) Matrix of zeros of size \( M \times N \)

\( (\cdot)^T \) transpose

\( (\cdot)^H \) Hermitian

\( (\cdot)^* \) conjugate

\( \otimes \) Kronecker product

\( \odot \) Hadamard product

\( \oslash \) Hadamard division

\( \lfloor \cdot \rfloor \) round down to integer

\( \text{diag}\{\cdot\} \) generate a block diagonal matrix

\( \|\mathbf{a}\| \) standard norm of the column vector \( \mathbf{a} \)

\( \mathbf{a}^n \) element-wise

\( \mathcal{R} \) field of real numbers

\( \mathcal{C} \) field of complex numbers

\( \mathcal{Z} \) group of integer numbers

\( \mathcal{I} \) field of imaginary numbers

\( \dot{x}_p \) partial derivative of \( x \) with respect to the variable \( p \)

\( \|\mathbf{A}\|_F \) Frobenius norm of the matrix \( \mathbf{A} \)
Chapter 1

Introduction

THE evolution of wireless communications systems has always been driven by the need for greater information data rates, a need arising from the ever more demanding end-user applications. Given the cost and scarcity of the available radio spectrum, engineers struggle to design spectrum efficient wireless communications systems by devising more sophisticated tranceivers and signal processing algorithms. The introduction of Turbo codes in 1993 [3] allowed for systems operating only a fraction of a dB away from the Shannon capacity limit. With no significant margin for improvement by using more advanced Error Correction Coding (ECC) techniques, the current 3G and under development post-3G standards involve (or are likely to involve) novel technologies. Among others, Orthogonal Frequency Division Multiple Access (OFDMA) and Multicarrier Code division Multiple Access (MC-CDMA) have been either proposed or incorporated into the latest and future standards, for example W-CDMA, the air interface of UMTS, as well as in 802.16e (WiMAX) and 802.20 standards. In addition, most of these schemes support or demand, the use of multiple antennae at both the receiver and transmitter side of the radio link.

Multiple-input, multiple-output (MIMO) systems allow for the use of space-time signal processing on both ends of the communication link. They have enjoyed wide attention from the scientific community because of their potential to increase the overall channel capacity and link quality without added transmission power or extra bandwidth requirements. However, most of the research on MIMO systems
ignores the spatial configuration of the array and the resulting spatial correlation of the received signals, induced by the geometry of the antenna array.

Systems which take into account the geometry of the antenna array (henceforth array systems in order to distinguish them from conventional MIMO systems) have been used in a wide spectrum of commercial and scientific applications; wireless communications systems, radar and sonar systems, surveillance and localisation systems, seismology, astronomy, medical tomography [55]. Recently, array systems have been used in the area of wireless sensor networks [16, 29] of small sensor nodes.

All the characteristics of an array system can be incorporated in the array manifold, which represents the response of an array system. In mathematical terms, the array manifold is a geometric object, embedded in a multidimensional complex space. Thus, the main objective of this research thesis is to investigate the properties of these geometric objects, which are widely used in array signal processing algorithms employed in array systems. What has to be pointed out is that this investigation is carried out in an abstract level and is not, at least initially related to a specific array system or any specific array application.

The rest of this chapter is organised as follows: in Section 1.1 the array manifold is defined and basic notions of Differential Geometry employed in the rest of this thesis are introduced. Next, in Section 1.2, a comprehensive exposition of the most important research work to date on the array manifold will be presented. Finally, in Section 1.3 the various research questions tackled in the following chapters and the organisation of this thesis are outlined.

1.1 The Concept of the Array Manifold

Array processing and communications techniques, exploiting the structure of an antenna-array system, have evolved into a well-established technology, moving from old conventional direction finding and phased-arrays to arrayed MIMO systems, arrayed wireless sensor networks and advanced super-resolution space-time
array systems. Nevertheless, array systems demand efficient, robust and sophisticated algorithms, in order to reach their peak performance. For instance, a powerful class of algorithms which fits the above-mentioned criteria is the subspace-type algorithms. These algorithms are robust to near-far effects (i.e. they remove the need for power control) and can be used in a variety of signal environments, such as in synchronous and asynchronous, single and multicarrier CDMA communication systems.

These algorithms are based on the concept of the array manifold, which is the locus of all the array response vectors (manifold vectors) and maps the geometrical aspects of the array system to the signal environment. For an omnidirectional array of \( N \) elements, the array manifold is formally defined as

\[
\mathcal{M} \triangleq \{ \mathbf{a}(p, q) \in \mathbb{C}^N, \forall (p, q) : (p, q) \in \Omega \}
\]  

(1.1)

where \( p, q \) are directional parameters (e.g. azimuth \( \theta \) and elevation \( \phi \) angles of arrival as depicted in Fig. 1.1), \( \Omega \) is the parameter space, \( c \) is the velocity of the propagation of the electromagnetic wave and

\[
\mathbf{a}(p, q) = \exp \left( -j \frac{2\pi F_c}{c} \mathbf{r} \cdot \mathbf{u}(p, q) \right)
\]

(1.2)

is the manifold vector, where \( \mathbf{r} = [r_x, r_y, r_z] \) is the matrix containing the \( x, y, z \) coordinates of the sensor elements in units of meters, \( F_c \) is the carrier frequency and \( \mathbf{u}(p, q) \) is the unit vector pointing towards the direction \( (p, q) \).

The array manifold vector (see Fig. 1.1) models the response of an array of omnidirectional elements, in the case of a unity powered signal impinging on the array from direction \( (p, q) \), with respect to the array reference point and the chosen system of coordinates. It should be noted that plane wave propagation is assumed for the derivation of Equation (1.2). Furthermore, the signal is assumed narrowband, so that the baseband information signal does not change significantly within the time required for the plane wave to traverse the antenna array, i.e.
1.1 The Concept of the Array Manifold

\[ m(t + \Delta \tau) \cong m(t) \], where

\[ \Delta \tau = \max_{k,l} \left| \frac{(r_k - r_l)^T u}{c} \right| \]

and \( c \) is the propagation velocity of the radio signal. Thus, the differences in the measured signals at the array reference point and at the \( k \)-th sensor, \( k = 1, \ldots, N \) is due to the rotation \( \omega_c \Delta \tau_k \) of the phasor of the incoming signal.

An equivalent formulation of Equation (1.2) is the following.

\[ \bar{a}(p, q) = \exp \left( -j\pi \left[ r_x, r_y, r_z \right] u(p, q) \right) \]  (1.3)

in which the matrix of the array elements coordinates \( [r_x, r_y, r_z] \) is in units of half-wavelengths. For the rest of this thesis, unless otherwise stated, the units of the sensor coordinates will be considered to be half-wavelengths.

The array manifold vector of Equation (1.3) reflects changes in the measured signal which depend exclusively on the geometry of the antenna array and the direction of arrival of the incoming signal. In the related literature, the object defined in Equation (1.3) is referred to as the array manifold vector. However, as it will be shown in Chapter 2, it is possible to define new array manifold vectors, which will again model the response of various array systems. Hence, in the rest of this thesis, the terms spatial array manifold (or simply array manifold) and spatial array manifold vector (or simply array manifold vector) will be used whenever reference to Equations (1.1) and (1.3) is required. The terms extended array manifolds and extended array manifold vectors will be used within this thesis to refer to all existing and future definitions of mathematical objects emerging as the locus of various array response vectors.
Figure 1.1: Physical interpretation of the array manifold vector of an array of omnidirectional elements in the case of a far-field, narrowband signal
1.2 Previous Work on the Geometry of the Array Manifolds

Existing research has focused exclusively on the investigation of the geometric properties of (spatial) array manifolds embedded in $N$-dimensional complex spaces. However, the spatial array manifold is a member of the set of the extended array manifolds and all extended array manifolds will be shown to represent geometric objects embedded in multidimensional complex spaces. Hence, the geometric concepts presented next will serve as an introduction to the basic notions of Differential Geometry, notions indispensable for the study of any extended array manifold.

The spatial array manifold is a multidimensional complex non-linear subspace in $\mathbb{C}^N$, or equivalently a multidimensional real subspace of $\mathbb{R}^{2N}$ equipped with the standard complex structure [19]. The term “manifold” is mathematically defined as a topological space which satisfies the following conditions:

1. It is second countable, i.e. it can be described by a countable basis.

2. It is a Hausdorff space, i.e. its points can be separated by neighbourhoods.

3. It is locally homeomorphic to a Euclidean space.

Since, by its definition,

$$\bar{a}^H a = N$$

the spatial array manifold is a non-linear subspace of the $N - 1$-dimensional complex sphere with radius equal to $\sqrt{N}$ [21].

Equations (1.1) and (1.3) define a regular parametric representation (in terms of the directional channel parameters $p$ and $q$) of a geometric object embedded in an $N$-dimensional complex space. If the spatial array manifold depends solely on one parameter, it represents a curve; if it depends on two parameters (e.g. azimuth/elevation or azimuth/frequency), it represents a surface.
In general, the extended array manifolds, which will be presented in Chapter 2, will be a function of a number of channel parameters, such as the bearing parameters $p$ and $q$ of the incoming signal, the delay of each incoming path, the PN-code of each user in a CDMA system, the polarisation of the electromagnetic wave, the doppler effects, etc. This implies that the array manifolds will, in the most general case, represent geometric objects of dimensionality greater than two, namely “hypersurfaces”. However, there is a multitude of theoretical results and techniques for studying curves (1-dimensional objects) and surfaces (2-dimensional objects) embedded in multidimensional real spaces, dating back to the work of Gauss and Riemann. In addition, the spatial array manifold of Equation (1.1) depends (apart from the array geometry) on the directional characteristics of the incoming signal. Hence, it is at most a 2-dimensional object, since far field sources are assumed. Thus, as a natural consequence, existing studies on the properties of the array manifolds have focused on array manifold curves and surfaces only. The same approach will be used herein, in order to take advantage of the huge mathematical arsenal on the treatment of 1 and 2-dimensional geometric objects. The rest of the channel parameters affecting the array manifolds will be considered to be fixed, which implies that a multidimensional “slice” of the array manifold, the one corresponding to parameters of interest, is investigated.

1.2.1 Spatial Array Manifold Curves

The geometric properties of the spatial array manifold curve have been extensively studied in the literature [7,31,48]. If the spatial array manifold is a function of one parameter only, it corresponds to a curve embedded in a complex $N$-dimensional space. In this case, Equation (1.1) becomes

$$\mathcal{A} = \{ \mathbf{a}(p) \in \mathbb{C}^N, \forall p : p \in \Omega \}$$

(1.4)
where \( p \) may be the azimuth angle \( \theta \), the elevation angle \( \phi \) or any other equivalent channel parameter of interest, which, however, has to be related to the free variable (\( \theta \) or \( \phi \)) via an allowable change of parameters.

The spatial array manifold curve, like any array manifold curve, can be treated as a curve embedded in a complex \( N \)-dimensional space. In this case, the most important properties of the array manifold curve at some point \( a(p) \) are

1. the tangent vector \( \dot{a}_p(p) \triangleq \frac{\partial a}{\partial p} \);
2. the arc length \( s(p) \);
3. the rate of change of the arc length \( \dot{s}(p) \);
4. the set of coordinate vectors \( u_k(p) \), \( k = 1, \ldots, d \) at a point \( p \) of the manifold curve;
5. the set of curvatures of the manifold curve \( \kappa_k(p) \), \( k = 1, \ldots, d \), which form the Cartan Matrix \( C(p) \)

\[
C(p) = \begin{bmatrix}
0 & -\kappa_1 & 0 & \ldots & 0 & 0 \\
\kappa_1 & 0 & -\kappa_2 & \ldots & 0 & 0 \\
0 & \kappa_2 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -\kappa_{d-1} \\
0 & 0 & 0 & \ldots & \kappa_{d-1} & 0 
\end{bmatrix}
\]

The arc length \( s(p) \) of the array manifold curve is formally defined as

\[
s(p) = \int_{p_0}^{p} \| \dot{a}_\xi(\xi) \| \, d\xi \quad (1.5)
\]

where \( p_0 \) is the value of the parameter \( p \) for which \( s(p_0) = 0 \). The arc length is an extremely important parameter because it is the most basic feature of a curve and a natural parameter representing the actual physical length of a segment of
1.2 Previous Work on the Geometry of the Array Manifolds

the manifold curve. Parametrization of the manifold curve $\mathcal{A}$ in terms of the arc length $s$ is more suitable in the treatment of space curves. Moreover, the arc length $s(p)$ is an invariant parameter, that is the tangent vector to the curve, expressed in terms of $s$, $\dot{a}'(s) \triangleq \frac{da(s)}{ds}$, has always unity norm [21].

The rate of change of the manifold length

$$\dot{s}(p) = ||\dot{a}_p(p)||$$

(1.6)

is a local property of the curve and is directly related to the resolution and detection capabilities of the array. The greater the rate of change of the manifold length at a point $\dot{a}(p)$ on the array manifold curve, the more accurate the estimation will be and the better the detection capabilities of the array at the corresponding bearing $p$, as will be properly shown in Chapter 3.

At any point $\dot{a}(p)$ of a curve embedded in $\mathcal{C}^N$ it is possible to attach a continuous, differentiable and orthonormal system of $2N$ coordinate vectors $u_k(p)$, $k = 1, \ldots, 2N$ [30, 31]. These coordinate vectors, the first two of which are depicted

![Graphical representation of the array manifold curve in $\mathcal{C}^N$](image)
in Figure 1.2, form a set of orthogonal, in the “wide”\(^1\) sense, unit vectors which are used to analyse the properties of the curve locally at each point \(s_o\). It is the main tool in the differential geometric treatment of curves as it is far easier and more natural to describe local properties (e.g. curvature, torsion) in terms of a local reference system than using a global one, like an orthonormal Euclidean coordinate system attached at the origin.

The curvatures of a space curve are of immense value in differential geometry because according to the fundamental uniqueness theorem [30] curvatures uniquely define a space curve expressed in terms of its arc length, except for its position in space. Moreover, the number of non-zero curvatures reveal symmetries in the geometry of the array manifold curve, since a curve with \(d - 1\), \(d \leq 2N\) non-zero curvatures is restricted to a \(d\)-dimensional subspace of \(\mathbb{R}^{2N}\). It can be proven that if the curvatures of the array manifold remain constant, then the shape of \(A\) will be a hyperhelix, embedded in an \(N\)-dimensional complex space. This property of manifold curves is of high importance, since based on this it can be proven [31] that the manifolds of the popular class of linear arrays (and also the elevation and \(\alpha, \beta\) curves of planar arrays), are curves of hyperhelical shape embedded in an \(N\)-dimensional complex space. Hyperhelical curves are easier to study and analyse due to the specific format of their expression, as will be shown in Chapter 4.

1.2.2 Spatial Array Manifold Surfaces

The spatial array manifold as it is defined in Equation (1.1) is a special case of extended array manifolds depending on two parameters of interest. Array manifolds that are functions of two free variables correspond to surfaces embedded in \(\mathbb{C}^N\). The most important geometric properties of space surfaces embedded in multidimensional complex spaces are the following [31,38]:

1. the manifold metric \(G(p,q)\);

\[^{1}\text{Re} \{z_i^H z_j\} = \delta_{ij}\]
2. the Christoffel matrices $\Gamma_1(p, q)$ and $\Gamma_2(p, q)$;

3. the Gaussian curvature $K_G(p, q)$;

4. the geodesic curvature $k_g$ of a curve on $M$.

and their physical interpretation is described below.

The manifold metric

$$G(p, q) = \begin{bmatrix} ||\dot{a}_p||^2 & \text{Re} \{\dot{a}_p^H \dot{a}_q\} \\ \text{Re} \{\dot{a}_q^H \dot{a}_p\} & ||\dot{a}_q||^2 \end{bmatrix}$$

(1.7)

is a $2 \times 2$ semi-positive definite symmetric matrix, with its elements known, in differential geometry terms, as the first fundamental coefficients (or metric coefficients). It is a function of the tangent vectors $\dot{a}_p$ and $\dot{a}_q$, which form a basis $T = [\dot{a}_p, \dot{a}_q]$ (1.8) of the tangent plane of the surface at the point $(p, q)$. The first fundamental coefficients provide a way of measuring trajectories on non-Euclidean spaces. Furthermore, $\sqrt{\det(G)}$ serves as a tool for detecting the changes of the shape of the manifold surface [21], while the area of a segment on the manifold surface can be calculated as

$$\int \int_{\omega} \sqrt{\det(G)} dpdq$$

The Christoffel symbol matrices of the first $\Gamma_1$ and second $\Gamma_2$ kind determine how the tangent plane varies as the point $(p, q)$ moves on the manifold surface. It is important to note that these matrices depend on the first fundamental coefficients and their derivatives only.

According to the Theorema Egregium by Gauss, to every point of a surface embedded in $\mathcal{R}^{2N}$ can be assigned a real number $K_g$, independently of any specific curve passing through it. This real number, the Gaussian curvature, provides an indication of the local shape of the surface at the neighborhood of that point. However, the array manifold surface is embedded in a real space of dimensionality
larger than 3, therefore the Gaussian curvature has to be defined in a way that it is independent of the normal plane at a point $g(p, q)$ of the manifold surface, since there is no unique tangent plane. Because of this, the definition of the Gaussian curvature given in [31] utilises the \textit{intrinsic} geometry of the manifold surface.

Finally, for curves lying on manifold surfaces two important parameters are the arc length $s$ and the geodesic curvature $\kappa_g$. The arc length of a curve has already been defined in the previous paragraph. The geodesic curvature $\kappa_g$ is defined using the first curvature $\kappa_1$ of the curve along the tangent plane to the surface at every point along the curve. By considering a curve on a surface connecting two points, the geodesic curvature assesses the similarity of this curve to a \textit{geodesic curve} (i.e. a curve which is equivalent to a straight line in Euclidean space).

However, the investigation of a surface as a mathematical object embedded in $N$-dimensional complex space presents many difficulties and complications. To overcome these obstacles, an alternative representation using two families of parameter curves has been proposed in [31]. These are the family of $p$-curves defined as

$$\{A_{pq_o}, \forall q_o : q_o \in \Omega_q\}$$

where $\Omega_q$ is the domain of $q$, and the family of $q$-curves defined in a similar fashion. Every value of the fixed parameter $q_o$ defines a new $p$-curve and the ensemble of $p$-curves provide a covering for the whole manifold surface. In terms of differential geometry, the original array manifold surface can be viewed as a \textit{fiber bundle}, where a $q$-curve is the \textit{base} space and the $p$-curves are the fibres or the other way around. By treating the manifold surface using the aforementioned two families of curves, then a unified framework for the analysis of linear and non-linear array geometries is possible and some representative results are summarized below [31].

- The manifolds of 3-dimensional arrays of arbitrary geometry are locally elliptic.
• The manifolds of 3D-grid arrays (for which $[r_x, r_y, r_z]^T [r_x, r_y, r_z] = cI_3$) are spherical.

• All planar arrays have a manifold of conoidal shape.

• If $(p, q)$ represent the azimuth and elevation angles respectively, then the family of elevation-curves is a family of geodesic curves (i.e. equivalent to straight lines in Euclidean space) for all grid and all planar arrays, while the family of azimuth-curves are not.

• The elevation-curves of 2-dimensional arrays are hyperhelices.

1.3 Organisation of the Thesis

The rest of this thesis is organised as follows.

Chapter 2 deals with the geometric properties of the extended array manifolds. Initially, the concept of the extended array manifold is introduced as a generalization of the conventional, spatial array manifold. This generic array manifold defines a class of array manifolds which includes all the array manifolds proposed in the literature so far. This generalization allows for the modelling of a large class of array systems, such as asynchronous, DS-CDMA systems, systems affected by Doppler effects, systems utilizing polarization sensitive elements, etc.

Next, the extended array manifold is treated as a geometric object embedded in a multidimensional complex space. This study focuses on extended array manifolds dependent on one or two parameters only, that is extended array manifold curves and extended array manifold surfaces respectively. For both curves and surfaces, the geometric properties which were presented in the introduction are analytically derived. Finally, the general analytical formulae for the geometric properties of the extended array manifold are applied to estimate the geometric properties of each one of the array manifolds that have been introduced in the literature and were shown to be special cases of the generic concept of the extended array manifold.
In Chapter 3 the issue of theoretical performance bounds for array systems is treated. At first, the detection and resolution bounds defined in the literature for linear array systems are extended for array systems of any 3-dimensional geometry. These bounds were derived based on the concept of the spatial array manifold. Therefore, next these bounds are generalised for array systems being modelled by extended array manifolds and comparative studies are conducted for the performance of the various array systems under investigation.

Finally, a new approach is presented for determining the aforementioned performance bounds. The procedure used so far relied heavily on the estimation of a specific manifold curve on the manifold surface. However, this method could not be readily applied in any case as in most situations a complex coordinates transformation was necessary in order to determine the parameters of this manifold curve. The approach proposed here results in much simpler algebraic expressions, which greatly reduce the computational burden on both man and computer and have a clear geometric interpretation. The resulting expressions are shown to produce identical results with the methods used so far both for linear and general 3-dimensional array geometries.

Next, in Chapter 4 the study focuses on the concept of hyperhelical array manifold curves. Hyperhelical array manifold curves constitute a special class of array manifold curves, the defining characteristic of which is their constant curvatures throughout the length of the curve. This important characteristic facilitates the analysis of their geometry and consequently the computation of the theoretical performance bounds presented in Chapter 3. This chapter aims at determining which array geometries may be modelled by hyperhelical manifold curves. Thus, firstly, the general equation for a hyperhelical space curve in a multidimensional complex space is derived. Then, the corresponding regular parametric representation of a hyperhelical array manifold curve is established and based on this the respective classes of array geometries are identified.

Geometrical array uncertainties are the main topic of Chapter 5. Geometrical
uncertainties severely hinder the performance of array systems and they can be mitigated in two stages. First, robust array geometries can be designed, which will be affected by uncertainties as little as possible. Secondly, calibration algorithms can be applied to the array system, so that the uncertainties are removed. The first part of the chapter deals with modelling the effect of uncertainties in channel estimation methods using an approach based on the geometric properties of the array manifold. The results can be used to design array geometries robust to the effects of geometrical uncertainties.

Next, a novel blind calibration algorithm is proposed, which aims in mitigating the effects of geometrical uncertainties. The algorithm makes use of a single moving source and achieves a significant reduction in the position errors of the array elements. In addition, it improves the accuracy of the Direction-of-Arrival estimation method which is initially affected by both uncertainties and the movement of the source.

Finally, in Chapter 6 the most important results of this thesis are iterated and some suggestions for continuation of the work are given.
Chapter 2

Extended Array Manifolds

The spatial array manifold vector for isotropic sensors, as defined in Equations (1.2) - (1.3), is a function of the bearings of the incoming signals and the array geometry. In more sophisticated array systems, the modelling may include various additional parameters - variables, which are not included in Equations (1.2) - (1.3). Thus, a number of extensions of the spatial array manifold vector have been introduced in the literature;

- the Spatio-Temporal ARay manifold vector (STAR) [33], which is suitable for modelling asynchronous, DS-CDMA array systems;
- the Doppler-STAR manifold [40], which, additionally, takes into account any doppler phenomena present in the signal environment;
- the Polar-STAR manifold [35] which assumes that the array sensors are sensitive to the signal polarisation;
- the MC-STAR [46] manifold which can model asynchronous, MC-CDMA array systems.

All the aforementioned extensions of the spatial array manifold are themselves geometric objects, embedded in appropriate multidimensional complex spaces. Therefore, the study of these manifolds, under the general framework developed for the study of the spatial array manifold, can provide significant insight for the properties of each of the corresponding array systems. The goal of this chapter is
to introduce a generic model for the existing and, possibly, for future extensions of the array manifold vector, which will allow for a comprehensive study of the geometrical properties of these objects without having to analyse each one of them separately.

This chapter is organised as follows. First, in Section 2.1 some examples of extended array manifolds, such as the Spatio-Temporal ARray (STAR) manifold, are presented and a generic model of extended array manifolds is defined. In Section 2.2, the theoretical framework of the extended array manifolds is developed and the connection of the spatial and the extended array manifolds is investigated. Then, in Sections 2.3 and 2.4 the geometric properties of the extended array manifold curves and surfaces, respectively, are derived as a function of the corresponding properties of the spatial array manifold curves and surfaces.

2.1 Beyond the spatial array manifold

2.1.1 Examples of extended manifold vectors

In the previous section, a number of extensions of the spatial array manifold vectors have been introduced, aiming at modelling more complex signal environments and more sophisticated array systems. These new array manifolds can be seen as members of a new class of array manifolds which, henceforth, will be referred to in this thesis as extended array manifolds. Equations (2.1) - (2.4) provide representative examples of extended array manifold vector and are followed by a brief description of their associated array systems.

\[ h^{\text{STAR}}(p, q, l) = a(p, q) \otimes (J^l) \]  
\[ h^{\text{MC-STAR}}(p, q, l, F_k) = a(p, q) \otimes (J^l [l, F_k]) \]  
\[ h^{\text{POL-STAR}}(p, q, l, \gamma, \eta) = a(p, q) \otimes q(p, q, \gamma, \eta) \otimes (J^l) \]  
\[ h^{\text{DOP-STAR}}(p, q, l, f_D, n) = a(p, q) \otimes (J^l \otimes F_D [n]) \]
2.1 Beyond the spatial array manifold

where $a(p, q)$ is the spatial manifold vector of Equation (1.3) and

$$
J = \begin{bmatrix}
0_{2N_c-1} & 0 \\
I_{2N_c-1} & 0_{2N_c-1}
\end{bmatrix}
$$

(2.5)

is the downshifting matrix.

The STAR manifold vector of Equation (2.1) has been defined in [33] for the modelling of an asynchronous, CDMA array system. This new manifold vector is a function of the geometry of the array, which is included in the spatial manifold vector $a(p, q)$, a PN sequence of length $N_c$ contained in the $2N_c$ vector $\gamma$ (which is padded with $N_c$ zeros at the end) and the lack of synchronisation

$$
l = \left\lfloor \frac{\tau_{ij}}{T_c} \right\rfloor \mod N_c
$$

(2.6)

where $l$ is the discretised delay $\tau_{ij}$ of the $j$-th path of the $i$-th user in units of the code chip period $T_c$. The main advantage of the STAR manifold over the spatial array manifold is that it extends the degrees of freedom (and thus the observation space) from $N$ to $2NN_c$ and that allows for the handling of more incoming signals/paths than the number of the array elements.

Multicarrier CDMA systems are a strong candidate for the next generation wireless communications systems, because of their inherent ability to tackle frequency selective channels, while at the same time enjoying the advantages of spread-spectrum systems. For an array system where multicarrier transmission is used, the MC-STAR manifold vector was introduced in [46] and given in Equation (2.2), where

$$
a[l, F_k] \triangleq \gamma[k]
$$

and $N_{sc}$ is the number of different sub-carriers used in the multicarrier modula-
tion, \( F_k = (k - 1) \Delta f \) is the frequency corresponding to the \( k \)-th subcarrier, \( T_s \) is the sampling period, \( \Delta f \) is the separation in Hz between the subcarriers and \( \gamma[k] \) is the \( k \)-th element of a second PN-sequence \( \gamma \in \mathbb{Z}^{N_{sc}} \) used to spread the data among the subcarriers.

Polarisation information has been used in the literature to improve the detection and estimation capabilities of the array [35] and to deal with correlated [18] or coherent [28] sources. In order to model the effect of the signal polarisation on the channel, the Polar-STAR manifold vector of Equation (2.3) was introduced in [41], where

\[
q(p, q, \gamma, \eta) = V \left[ \cos \gamma \sin \gamma e^{\eta} \right]^T \tag{2.7}
\]
models the effect of a signal with a state of polarisation \((\gamma, \eta)\), with the \( i \)-th column of \( V \) representing the complex voltages induced at the \( i \)-th sensor in response to unit electric fields polarised in the x, y and z directions respectively (see [35]).

Finally, in [40] the authors presented an array system affected by Doppler effect and defined the Doppler-STAR manifold vector of Equation (2.4), where

\[
\mathcal{F}_D [n] = \begin{bmatrix}
1 \\
\exp (j2\pi f_D T_c) \\
\vdots \\
\exp (j2(N_c - 1)\pi f_D T_c)
\end{bmatrix}
\exp \{ j2\pi n f_D T_{cs} \}
\]
and \( f_D \) is the Doppler spread affecting the signal. Notice that the non-stationarity of the channel is reflected on the dependence of the Doppler-STAR manifold vector on the time index \( n \).

Each of the extended array manifold vectors of Equations (2.1) - (2.4) gives rise to the associated extended manifold \( \mathcal{H} \), which (similarly to the definition of the spatial array manifold as the locus of the spatial array manifold vector as \( p \)
and \( q \) vary) is defined as

\[
\mathcal{H} \triangleq \left\{ h(p) \in \mathbb{C}^Q , \forall \underline{p} : \underline{p} \in \Omega_p \right\}
\]  

(2.8)

where \( Q \in \mathbb{N} \) is the dimensionality of the complex space in which \( \mathcal{H} \) is embedded and \( \underline{p} \) represents the parameters upon which each respective extended manifold vector depends. For example, for the MC-STAR manifold, \( Q = 2NN_cN_{sc} \) and \( \underline{p} = [p, q, l, \Delta f]^T \).

### 2.1.2 Basic array system

Let us consider an array system of \( N \) omni-directional antennas operating in the presence of \( M \) co-channel users. There are no multipaths, so that only \( M \) signals arrive at the antenna array. The architecture of such a system is depicted in Figure 2.1. The \( i \)-th user transmits a modulating information signal:

\[
m_i(t) = \sum_{n=-\infty}^{\infty} a_i[n] c(t - nT_{cs})
\]  

(2.9)

where \( \{a_i[n], \forall n \in \mathbb{Z}\} \) represent the \( i \)-th user’s information symbols and \( c(t), t \in [0, T_{cs}) \) is the symbol pulse-shaping waveform of duration \( T_{cs} \).

![Figure 2.1: Conventional Array System Architecture](image)

The received signal \( \underline{x}(t) \) at the output of the array (Point A in Figure 2.1)
can be modelled as
\[ x(t) = S(p) \ B \ m(t) + n(t, \sigma_n^2) \] (2.10)
where \( x(t) \in \mathbb{C}^N \) is the vector of the observed signals and \( p \) denotes all the parameters of interest of all users, that is
\[ p = [p_1^T, \ldots, p_M^T]^T \]
with \( p_i, i = 1 \ldots M \) representing the parameters associated with the \( i \)-th user.

Furthermore
\[ S(p) = [a_1(p_1), \ldots, a_M(p_M)] \in \mathbb{C}^{N \times M} \]
is the matrix of the channel vectors (spatial manifold vectors) for each one of the \( M \) incoming signals and
\[ B = \text{diag} \{[\beta_1, \ldots, \beta_M]\} \] (2.11)
with \( \beta_i \) being the complex fading coefficient for the \( i \)-th incoming signal. Moreover
\[ m(t) = [m_1(t), m_2(t), \ldots, m_M(t)]^T \] (2.12)
is the vector containing the baseband information signals of the \( M \) sources at the array reference point and \( n(t, \sigma_n^2) \) is a \( N \times 1 \) complex vector of additive, zero mean, white gaussian noise, with covariance matrix equal to \( \sigma_n^2 I_N \).

At the receiver, the continuous signal vector \( x(t) \) is sampled at a rate \( T_s = T_{cs} \) to produce the observation vectors \( x(t_k), t_k = kT_s, k = 1, \ldots, L_{cs} \), where \( L_{cs} \) is the number of snapshots. Under these assumptions, the discretised output of the array system ( Point B of Figure 2.1) at \( t_k = kT_{cs}, k = 1, \ldots \) can be modelled as
\[ x(t_k) = S(p) \ B \ m(t_k) + n(t_k, \sigma_n^2) \] (2.13)
where \( m(t_k) \) is the vector given by Equation (2.12) (see also (2.9)) at time instant \( t = t_k \).

For the remaining of the thesis the system architecture of Figure 2.1 will be
referred to as the basic array system and it will be used for the analysis of more complex array systems.

### 2.1.3 Extended array systems

Figure 2.2 shows an alternative and more complex array communications system. The information symbols $a_i[n]$ of the $i$-th user, with a data symbol rate of $R_{cs} = T_{cs}^{-1}$, are fed into a Transmitter Linear Preprocessing Unit (Tx-LPU) which outputs the new symbols $b_i[k]$ with a symbol rate of $R_c = T_c^{-1}$, where $T_c$ is now the duration of the new data symbol pulse-shaping waveform.

At the receiver side, the continuous received vector signal $\mathbf{g}(t)$ at Point A in Figure 2.2 is sampled at a rate of $R_c$ to produce the sample vectors $\mathbf{g}(t_k), t_k = kT_c$ at Point B. These vectors are then fed into a Receiver Linear Preprocessing Unit (Rx-LPU), which produces, at Point C, the observation signal vectors $\mathbf{g}[n] \in \mathbb{C}^Q, Q \geq N$ at a rate $R_{cs}$.

The architecture shown in Figure 2.2 can be used as a generic representation of all systems described in Section 2.1.1. For example, the spreading of the data symbols to produce the CDMA chips, as well as the use of multiple carriers are linear processes and thus can be represented abstractly by the Tx-LPU. In addition, Doppler effects can be part of the wireless channel. Finally, the Rx-LPU may involve a Tapped Delay Line which is associated with extended array manifold vectors of Equations (2.1) - (2.3). Note that if the TDL acts on a batch of $2N_c$ vectors $\mathbf{g}(t_k)$, then this is a linear operation and its output $\mathbf{g}[n]$ can be mathematically expressed as follows

$$\mathbf{g}[n] = \sum_{k=(n-1)N_c+1}^{(n+1)N_c} \mathbb{P}(k \mod 2N_c) \mathbf{g}(t_k)$$ (2.14)
Figure 2.2: Extended Array System Architecture
where

\[ P_m = \begin{bmatrix} P_{m,1} \\ \vdots \\ P_{m,N} \end{bmatrix}, \quad P_{m,i} = P (J_N^T)^{i-1} \]

\[ P = \begin{bmatrix} 0_{m-1} & 0 \begin{pmatrix} (m-1) \times (N-1) \end{pmatrix} \\ 1 & 0_{N-1}^T \\ 0_{2N_c-m} & 0 \begin{pmatrix} (2N_c-m) \times (N-1) \end{pmatrix} \end{bmatrix} \]

and \( J_N \) is defined as

\[ J_N = \begin{bmatrix} 0_{N-1}^T & 0 \\ I_{N-1} & 0_{N-1} \end{bmatrix} \] (2.15)

The output \( \varphi(t_k) \) of the array system depicted in Figure 2.1, as can be seen from Equation (2.13), is a function of the spatial array manifold, which contains all the information regarding the array system. However, the output \( \varphi[n] \) of each of the four systems of Section 2.1.1 is a function of its associated extended array manifold.

In addition, the Transmitter-LPU and the Receiver-LPU, are not limited to the operations described earlier and may model any linear operation performed on the data symbols \( a_i[n] \) at the transmitter and on the sample vectors \( \varphi(t_k) \) at the receiver. Thus, the system architecture of Figure 2.2 may be used to model a larger class of array systems than the four described in Section 2.1. Hence, a class of extended array manifolds can be defined, which will consist of the extensions of the spatial array manifold required to model the output of an array system fitting the model of Figure 2.2. All of the array manifolds defined so far, including the spatial array manifold, are then members of this class.

Thus, the study of all the existing and, potentially, new array manifolds defined in the future is reduced to the study of the properties of the class of extended array manifolds. The nature of these properties, which will reveal the relation between the spatial and the extended array manifolds, is investigated in the following section.
2.2 Extensions of the spatial array manifold

All of the extended array manifold vectors presented in Section 2.1 are functions of the spatial manifold vector \( \mathbf{a} = \mathbf{a} (p, q) \). In detail, for reasons that will be made clear in this section, any extended array manifold can be viewed as the result of a complex mapping, satisfying certain constraints and acting on the spatial array manifold. The nature of these constraints, as well as the general properties of these mappings will be investigated next.

2.2.1 The extended manifolds as complex mappings

Let us consider an analytic complex mapping \( T : \mathcal{C}^N \rightarrow \mathcal{C}^Q \). An “analytic mapping” \( T \) is a vector function, every component of which, \( f_q : \mathcal{C}^N \rightarrow \mathcal{C}, q = 1, \ldots, Q \), is an analytic function. Applied on the spatial manifold vector \( \mathbf{a} \), this mapping produces a new vector \( \mathbf{h} \in \mathcal{C}^Q, Q \geq N \), that is

\[
\mathbf{h} = T (\mathbf{a}) = [f_1 (\mathbf{a}), f_2 (\mathbf{a}), \ldots, f_Q (\mathbf{a})]^T \quad (2.16)
\]

Equation (2.16) may assume a more specific format if one considers the nature of the various spatial manifold vector extensions. Based on Figure 2.2, these extensions arise either due to a change in the actual wireless channel, for example the introduction of Doppler effects, or because of the Tx-LPU and the Rx-LPU. The basic difference is that extensions due to a change in the wireless channel do not affect the dimensionality of the array manifold vector, simply because the dimensionality of the observation vectors at point B of Figure 2.2 remains unchanged. However, the insertion of the Rx-LPU on the receiver side (in order to model changes such as the co-existence of multiple information symbols in the TDL) and the effect of polarisation sensitive sensors do affect the dimensionality of the resulting the observation space and, thus, of the array manifold vector.

As the Tx-LPU and the Rx-LPU are linear blocks/systems and the wireless channel is assumed to be linear (i.e. FIR structure), then \( T \) will be a linear
mapping too. Hence, Equation (2.16) can be rewritten as

$$h(p) = A(p) a(p, q)$$

(2.17)

Note that in all the array systems presented in Section 2.1 the Rx-LPU consists of a bank of \( N \) parallel processing units, each one of which provides \( Q/N \) outputs. In this case the mapping \( T \) can be written as

$$h = [f_1(a_1), f_2(a_1), \ldots, f_q(a_k), \ldots, f_Q(a_N)]^T$$

(2.18)

where all \( f_q(a_k) \) is a linear function of the \( k \)-th element \( a_k \) of the spatial array manifold vector \( \mathbf{a} \). Thus, Equation (2.17) can be reformulated as

$$h(p) = A(p) a(p, q)$$

(2.19)

with \( A(p) \) having only one non-zero element in each row, which implies that

$$A^H A \in \mathbb{R}^{N \times N} \text{ and diagonal}$$

(2.20)

Finally, as long as the \( N \) preprocessors are identical, that is that the output of all the array elements is affected in a similar way, then \( A \) can be expressed in the following format

$$A(p) = I_N \otimes \tilde{z}(p)$$

(2.21)

where

$$\tilde{z}(p) \in \mathbb{C}^Q, \frac{Q}{N} \in \mathbb{N}$$

(2.22)
2.3 Extended Manifold Curves

2.3.1 Basic theoretical framework

The extended array manifolds are geometric objects embedded in multidimensional complex spaces. In this section, manifolds which depend on one parameter only (i.e. curves) will be considered. Since the investigation of the spatial array manifold has focused on the bearing curves (i.e. azimuth, elevation, directional cosine curves), the parameter \( p \) in the following analysis will stand for a directional parameter, which implies that the vector parameter \( p \) reduces to the scalar parameter \( p \). In this case, the locus of all extended manifold vectors \( \mathbf{h}(p) \) defined by Equation (2.19) is given by

\[
\mathcal{H}_c \triangleq \{ \mathbf{h}(p) \in \mathbb{C}^Q, \forall p : p \in \Omega_p \} \tag{2.23}
\]

Equation (2.23) defines a curve embedded in a \( Q \)-dimensional complex space. Note that the dependence on the fixed parameters (such as the array geometry, the frequency of the carrier, etc.) has been dropped, for simplicity.

The tangent vector at each point \( p \) of the extended manifold curve is given by

\[
\dot{\mathbf{h}}(p) \triangleq \frac{\partial \mathbf{h}(p)}{\partial p} = \frac{\partial \mathbf{A}(p)}{\partial p} \mathbf{a}(p) + \mathbf{A}(p) \frac{\partial \mathbf{a}(p)}{\partial p} \tag{2.24}
\]

For the sake of simplicity, the dependence of the various vectors and matrices on the variable \( p \) will be dropped and only explicitly mentioned when it is needed. Moreover, \( \dot{x} \) will denote differentiation of \( x \) with regards to the variable \( p \).

The rate of change of the arc length of \( \mathcal{H}_c \) is

\[
\dot{s}_{\mathcal{H}_c}(p) = \left\| \dot{\mathbf{h}}(p) \right\| = \
\sqrt{\dot{\mathbf{a}}^H \dot{\mathbf{A}}^H \dot{\mathbf{a}} + \dot{\mathbf{A}}^H \dot{\mathbf{A}} + 2 \text{Re} \left\{ \dot{\mathbf{a}} H \dot{\mathbf{A}} \dot{\mathbf{a}} \right\}} \tag{2.25}
\]
However

\[ \begin{align*}
\dot{A}^H \dot{A} &= \frac{||\dot{z}(p)||^2}{\sigma^2_A(p)} \mathbb{I}_N, \\
\ddot{A}^H \ddot{A} &= \frac{||\ddot{z}(p)||^2}{\sigma^2_{\dot{A}}(p)} \mathbb{I}_N \\
\ddot{A}^H \dot{A} &= \ddot{z}^H \ddot{z}^A \dot{A} = 0
\end{align*} \]

(2.26)

where \( \ddot{A}^H \dot{A} = 0 \) because of the fact that the array centroid has been taken as the array reference point (i.e. \( \frac{1}{N} \sum_{i=1}^{N} r_i = 0 \)). Based on Equations (2.26), Equation (2.25) may be rewritten as

\[ \dot{s}_{\mathcal{H}_c}(p) = \sqrt{\sigma^2_A(p) \dot{s}^2_A(p) + \sigma^2_{\dot{A}}(p) N} \]

(2.27)

where

\[ \dot{s}_A(p) \triangleq \sqrt{\dot{a}^H \dot{a}} = \pi \sin p \| \cdot \|_r \]

(2.28)

is the rate of change of arc length of the corresponding spatial manifold curve \( \mathcal{A} \) (see Equation (1.4)). It is important to note that the rate of change of arc length of the extended manifold curve \( \mathcal{H}_c \) has been expressed as a function of the corresponding geometric property of the spatial manifold curve and the scalar functions \( \sigma_A(p) \) and \( \sigma_{\dot{A}}(p) \) which code the properties of the mapping \( T \). Equation (2.27) provides a deeper insight into the relation between the spatial and the extended array manifold curves. The right hand side of Equation (2.27) consists of two terms; the first one of these terms is the square of rate of change of arc length of the spatial manifold curve \( \mathcal{A} \) scaled by the factor \( \sigma^2_A(p) \). The second term includes the factor \( ||\ddot{z}(p)||^2 \) which can be regarded as the rate of change of the arc length of a new manifold curve \( \mathcal{A}_\ddot{z} \), defined as

\[ \mathcal{A}_\ddot{z} \triangleq \left\{ \ddot{z}(p) \in C^N, \forall p : p \in \Omega_p \right\} \]

(2.29)

The multiplicative factors \( \sigma^2_A(p) \) and \( N \) act as weighting factors because of the different dimensionality of the embedding spaces for the two curves.

The total length of \( \mathcal{H}_c \) is given by

\[ \ell_{\mathcal{H}_c} = \int_{\Omega_p} \dot{s}_{\mathcal{H}_c}(p) \, dp \]

(2.30)
where $\Omega_p$ is the domain of $p$. The first coordinate vector, a unit vector co-linear with the tangent vector $\dot{h}(p)$ can be expressed as

$$u_1 = u_1(s) \triangleq \frac{d}{ds} = \frac{\dot{h}(p)}{\dot{s}_{H_c}(p)}$$

(2.31)

where $s$ is the total arc length of the manifold curve up to the point $p$ and it can be related to the bearing parameter $p$ via the *allowable change of parameter* [21] defined in Equation (2.32).

$$s_{H_c}(p) = \int_{p_0}^{p} \dot{s}_{H_c}(\xi) \, d\xi$$

(2.32)

Because the extended manifold curves are embedded in a $Q$-dimensional complex space, it is possible to attach to every point of the manifold curve up to $2Q$ coordinate vectors and $2Q$ curvatures, which form the Cartan matrix $C$ and uniquely define the shape of the manifold curve [21,30]. However, any symmetry in the geometry of the array system will reduce the dimensionality of the subspace of $C^Q$ in which the array manifold curve is embedded [31]. Therefore, in general up to $d$ coordinate vectors may be defined, where $\frac{d}{2}$ is the dimensionality of the complex subspace of $C^Q$ in which the array manifold curve lies.

These $d$ coordinate vectors and curvatures can be analytically calculated by applying the formulae presented in [31] to the model of the extended array manifolds presented earlier in this paper and making use of Equations (2.23) - (2.28). The derivation of all the coordinate vectors and subsequently the curvatures of the extended manifold curves, although analytically possible, is in practice extremely arduous. However, the most interesting differential geometric properties from an array processing point of view are

- the first and second coordinate vectors $u_1, u_2$;
- the rate of change of the arc length $\frac{ds}{dp}$;
- and the first curvature $\kappa_1$. 


These properties have been used extensively in the study of array systems, such as in determining the resolution and detection capabilities of linear [34], planar [32] and 3D [12] array geometries, the Cramer-Rao Lower bound for the estimation of channel parameters [32], analysing the ambiguities of linear and planar arrays [36] and finally in array design [11].

The second coordinate vector \( u_2(s) \) and the first curvature \( \kappa_1 \) can be calculated using the following equations

\[
\begin{align*}
  u_2(s) &= \frac{u_1'(s)}{\|u_1'(s)\|} \\
  \kappa_1(s) &= \|u_1'(s)\|
\end{align*}
\]  

(2.33) (2.34)

where \( x' \) denotes differentiation with regards to the natural parameter \( s \). Based on these definitions and on Equations (2.24) - (2.31) it is straightforward to show that the principal (first) curvature and the second coordinate vector are

\[
\begin{align*}
  \kappa_1(p) &= \frac{\|\varepsilon_1 d_o - d_1 e_o\|}{d_3^3} \\
  u_2(p) &= \frac{\varepsilon_1 d_o - d_1 e_o}{\|\varepsilon_1 d_o - d_1 e_o\|}
\end{align*}
\]  

(2.35) (2.36)

where

\[
\begin{align*}
  \varepsilon_o(p) &\triangleq \dot{h}(p) \\
  \varepsilon_1(p) &\triangleq \frac{\partial \dot{h}(p)}{\partial p} \\
  d_o(p) &\triangleq \ddot{s}_{H_o}(p_o) \\
  d_1(p) &\triangleq \ddot{s}_{H_o}(p)
\end{align*}
\]  

(2.37)

Based on the expression of the rate of change of the arc length of the extended array manifold curves, Equation (2.25), and the estimation of the first curvature given in Equation (2.35) we are now in position to compare the bearing curves of the various extended array manifolds presented in Section 2.1.1. Note that the analysis in the following section will be limited in the case of linear arrays or Equivalent Linear Arrays (ELA) of planar arrays for the shake of simplicity.
2.3 Extended Manifold Curves

2.3.2 Hyperhelical extended manifold curves

As it has been already said, in the most general case of the extended array manifold curves, the matrix of the linear mapping $A(p)$ depends on some fixed channel parameters and on the bearing variable $p$. However, if $A$ is independent of the bearing parameter $p$, then the resulting linear mapping has some extra structure. This structure enables for the analytical evaluation of the entire spectrum of the geometric properties of the extended manifold curve in terms of the corresponding properties of the spatial manifold curve and the matrix $A$. The detailed analysis of the curvatures of extended manifold curves, resulting from conformal\textsuperscript{1} linear mappings of the spatial array manifold curves will be presented in Appendix A. The most significant result, summarised in Theorem 1, will be given here and used to calculate the curvatures of various hyperhelical extended array manifold curves.

**Theorem 1.** Let $A \triangleq \{a(p) \in \mathbb{C}^N , \forall p : p \in [0,180^\circ]\}$ be the spatial array manifold curve of a linear array of $N$ elements, the shape of which is described by the Cartan Matrix $C(p)$, containing the spatial curvatures. Let another manifold curve $H_c \triangleq \{h(p) \in \mathbb{C}^Q , \forall p : p \in [0,180^\circ]\}$, related to $A$ via the linear mapping $T : \mathbb{C}^N \rightarrow \mathbb{C}^Q$. That is, $A(p) \in \mathbb{C}^{Q \times N}$ is the matrix representing the mapping $T$,

$$h(p) = A(p) \cdot a(p)$$

Assuming that $T$ (and as a result $A$) satisfy the assumptions of Section 2.2.1, if $\frac{\partial h(p)}{\partial p} = 0$ then $A^H A = \sigma^2 A I_N$ \textsuperscript{2} and the Cartan matrix $C_{H_c}(p)$ describing the shape of $H_c$ can be calculated as

$$C_{H_c}(p) = \frac{1}{\sigma_A} C(p) \quad (2.38)$$

Furthermore, the coordinate vectors $u_{H_c}(p), i = 1 \ldots, d$ of $H_c$ can be expressed as functions of the coordinate vectors $u_{A}(p), i = 1 \ldots, d$ of $A$ as follows

$$u_{H_c}(p) = \frac{1}{\sigma_A} A \cdot u_{A}(p) \quad (2.39)$$

\textsuperscript{1}A mapping is conformal when it preserves the angles between vectors

\textsuperscript{2}Note that now $\sigma_A$ is not a function of $p$
The proof of this Theorem and a derivation of closed form formulae for the expression of the coordinate vectors $\mathbf{u}_{i,\mathcal{N}_c}$, $i = 1, \ldots, d$ and the curvatures $\kappa_{i,\mathcal{N}_c}$, $i = 1, \ldots, d$ of the extended array manifold curves can be found in Appendix A.

The importance of this theorem lies in the fact that if the conformality assumption is satisfied, then the curvatures of the extended manifold curves are just a scaled version of the curvatures of the original, spatial array manifold curves. However, it is known from [31] that the curvatures of the spatial manifold curves of linear arrays are constant and independent of the bearing variable $p$, that is these curves are of hyperhelical shape. Thus, the extended manifold curves are also of hyperhelical shape. Therefore, the theoretical tools developed so far for the study of the hyperhelical manifold curves [31] can be readily applied to study the extended manifold curves as well.

Three of the extended array manifolds presented in Section 2.1 are produced by conformal linear mappings, the STAR manifold, the Doppler and Doppler-STAR manifolds and finally the Multi-Carrier-STAR manifold. Using Theorem 1 and the known curvatures of the corresponding spatial manifold curve, their curvatures can easily be estimated. The required $A$ and $\sigma_\Lambda$ parameters for these cases are as follows:

1. **STAR manifold curves**

   \[
   A^\text{STAR} = I_N \otimes \mathcal{J}_l^l, \\
   \sigma^\text{STAR}_\Lambda = \| \mathcal{J}_l^l \| = \begin{cases} 
   \sqrt{N_c}, & 0 \leq l \leq N_c \\
   \sqrt{2N_c - l}, & N_c < l < 2N_c 
   \end{cases} \tag{2.40}
   \]
2. Doppler-STAR Manifold curves

\[ A_{\text{DOP-STAR}} = I_N \otimes J_l \otimes E_D [n] \]

\[ \sigma_{A_{\text{DOP-STAR}}} = \| \mathcal{J}_l \| = \begin{cases} \sqrt{N_c} & , \ 0 \leq l \leq N_c \\ \sqrt{2N_c - l} & , \ N_c < l < 2N_c \end{cases} \quad (2.41) \]

3. Multi-carrier STAR Manifold Curves

\[ A_{\text{MC-STAR}} = I_N \otimes (\mathcal{J}_{l,a} [l, F_k]) \]

\[ \sigma_{A_{\text{MC-STAR}}} = \begin{cases} \sqrt{N_c N_{sc}} & , \ 0 \leq l \leq N_c N_{cs} \\ \sqrt{2N_c N_{cs} - l} & , \ N_c N_{cs} < l < 2N_c N_{cs} \end{cases} \quad (2.42) \]

where the various parameters have been defined in Section 2.1.1.

From Equations (2.40) - (2.41) it is clear that the parameter \( \sigma_A \) is the same for the STAR and the Doppler-STAR manifold curves. Consequently, the curvatures and, therefore, the shape of these manifold curves is identical. However, the orientations of the extended manifold curves within the \( 2NN_c \)-dimensional complex space are not identical and this clearly distinguishes the two manifold curves.

The curvatures for the spatial and the STAR, Doppler-STAR manifold curves for a symmetrical and a non-symmetrical linear array have been estimated and their values are plotted in Figures 2.3 - 2.4. Note that the tag “extended curve” in the graphs refers both to the STAR and the Doppler-STAR curves. The number of non-zero curvatures is the same for both the spatial and the extended hyperhelical manifold curves which arise from the same array geometry. This implies that both curves lie in a subspace of the overall observation space of the same dimensionality.

It is important to note that if the assumption that “the path-delay is smaller than the period of a data symbol \( T_{cs} \)” is invalid, or equivalently the discretised delay \( l \) is greater than \( N_c \), then only some of the energy of the current data
Figure 2.3: Curvatures of spatial and extended hyperhelical manifold curves, symmetrical array, sensor locations at $[-2, -1, 0, 1, 2], N_c = 15$

Figure 2.4: Curvatures of spatial and extended hyperhelical manifold curves, asymmetrical array, sensor locations at $[-3, -1, 0.9, 2.6], N_c = 15$
symbol is contained within the TDL (see Section 2.1 and [33]). The total energy of a symbol in the TDL is directly connected to the radius of the hypersphere on which the manifold curve lies. This is the reason why the curvatures of the extended manifold curves increase in magnitude as a function of $l$ as $l > N_c$.

### 2.3.3 Non-hyperhelical extended manifold curves

In this subsection, the results of Section 2.3.1 will be used to analyse the extended manifold curves of linear arrays which do not preserve the hyperhelical shape of the spatial manifold curves. Such non-hyperhelical manifold curves are:

1. The POLAR-STAR manifold curves.

#### POLAR-STAR Manifold curves

To study the POLAR-STAR manifold curves, the overall mapping will be partitioned into two mappings. The first part will provide an intermediate curve (POLAR curve) which will be studied using the proposed framework in Section 2.3.1. The second mapping, from POLAR to POLAR-STAR, will be shown to satisfy the assumptions of Theorem 1 and, thus, this theorem will be applied.

The POLAR manifold vector in the case of crossed-dipole arrays and for coplanar sources can be written as

$$h_{\text{POLAR}} = \left[ I_N \otimes q(p) \right] a(p)$$  \hspace{1cm} (2.43)

where $q(p)$ has been defined in Equation (2.7). Based on Equation (2.43) one may calculate for this intermediated curve

$$\mathbf{k}_{\text{POLAR}} = I_N \otimes q(p)$$

$$\sigma_{\mathbf{k}}^{\text{POLAR}} = \sqrt{V_x^2 \sin^2 p \cos^2 \gamma + V_z^2 \sin^2 \gamma}$$

$$\sigma_{\dot{\mathbf{k}}}^{\text{POLAR}} = V_x \cos p \cos \gamma$$  \hspace{1cm} (2.44)

with $V_x, V_z, \gamma$ defined in Section 2.1.1 and in [41].
Furthermore, Equation (2.3) of the POLAR-STAR manifold vector can be expressed as a function of $h_{\text{POLAR}}$ in Equation (2.43) as follows:

$$
\begin{align*}
h_{\text{POLAR-STAR}} &= [I_{2N} \otimes \mathbb{J}_2] h_{\text{POLAR}} \\
&\triangleq A
\end{align*}
$$

(2.45)

Since $A$ is independent of the bearing parameter $p$, Theorem 1 can be applied for the estimation of the geometric parameters of the POLAR-STAR curves based on those of the POLAR manifold curves.

**Manifold curves of arrays of directional elements**

Consider a linear or ELA having directional elements (ie non-isotropic). In this case, the array response vector ($\text{weighted}$ array manifold vector) can be expressed as a function of the array manifold vector $\mathbf{a}(p)$ for isotropic elements (see Equation (1.3)) as follows

$$
\mathbf{a}_w(p) = g(p) \odot \mathbf{a}(p) = A_w \mathbf{a}(p)
$$

(2.46)

where

$$
A_w \triangleq \text{diag} \{ g(p) \}
$$

(2.47)

and $g(p) \in \mathbb{C}^N$ is a vector containing the complex-valued gains of each array element. Of course, Equation (2.46) defines a linear mapping of the spatial manifold vector.

In Fig. 2.5 - 2.6 four different manifold curves of linear array have been studied. The first one is the spatial manifold curve of a Uniform Linear Array (ULA), the elements of which are positioned at $r_x = [−2, −1, 0, 1, 2]$ in units of half wavelengths. The second curve is the curve of the same array, with the exception that the elements are now considered to be sensitive to the polarisation of the incoming signal, with polarisation parameters equal to (2.43)

$$
V_x = 0.5, \ V_z = 0.5, \ \gamma = 0.1, \ \eta = 0.5
$$

(2.48)
The other two curves are the manifold curves of the same array, when the elements of the array are assumed to have the following gain patterns

\[ g_1(p) = \sin p + j \sin p, \quad g_2(p) = \frac{\sin \left(p - \frac{\pi}{2}\right)}{\left(p - \frac{\pi}{2}\right)} \]

The dependence of the principal curvature on the directional parameter \( p \) implies that specific operational characteristics of the array systems modelled by these manifolds are more sensitive of the direction of arrival of the signals. For example, the theoretical resolution threshold [31, 34] and the Cramer-Rao lower bound [31, 52] are dependent on the rate of change of the arc length, which is \( p \)-dependent even in the hyperhelical manifold curves, but also on the principal curvature.
Figure 2.5: Rate of change of the arc length of various extended manifold curves

Figure 2.6: Principal curvature of various extended manifold curves
In the previous section, the geometric properties of the extended array manifold curves have been studied as a function of the geometric properties of the spatial array manifold curves and the nature of the extension. The focus is now redirected to 2-dimensional extended array manifolds.

Assuming that the extended array manifold vector is now dependent on both bearing parameters $p$ and $q$, the extended array manifold is defined as

$$M_H \triangleq \{h(p,q) : (p,q) \in \Omega_{p,q}\}$$  \hspace{1cm} (2.49)

where, according to Equation (2.19),

$$h(p,q) = \mathbf{h}(p,q) \mathbf{a}(p,q)$$

The objective of this section is the investigation of the most important geometric properties of the extended array manifold surfaces, which were presented in Chapter 1. In detail, a link will be established between the properties of the extended array manifold surfaces and the corresponding properties of the spatial array surfaces (from now on extended and spatial surfaces respectively.) This link, will not only allow for an efficient computation of the geometric properties of the extended surfaces, given the existing results for the spatial manifold surfaces, but will also provide an insight into the geometrical differences and similarities between the two manifolds.

Note that, in the analysis that follows, every variable with a subscript $H$ will refer to the geometric properties of the the extended array manifold surface $M_H$, whereas variables without a subscript will refer to the properties of the spatial array manifold surface $M$, where

$$M \triangleq \{a(p,q) \in \mathcal{C}^N, \forall(p,q) : (p,q) \in \Omega\}$$  \hspace{1cm} (2.50)
2.4 Extended Array Manifold Surfaces

2.4.1 “Conformal” extended array manifold surfaces

Consider a 3-dimensional array of \( N \) omni-directional elements. Its response is described by the vector \( \mathbf{a}(p, q) \) given in Equation (1.3) and the spatial array manifold is a surface embedded in an \( N \)-dimensional complex space. Moreover, consider a linear mapping \( T : \mathbb{C}^N \rightarrow \mathbb{C}^Q \) (ie from an \( N \)-dimensional space to a \( Q \)-dimensional space), described by a matrix \( \mathbf{A} \), with

\[
\mathbf{A} = \mathbb{I}_N \otimes \mathbf{z}(p, q)
\]

according to Section 2.2.1. In this section, the case where

\[
\frac{\partial \mathbf{A}}{\partial p} = \frac{\partial \mathbf{A}}{\partial q} = 0
\]

will be investigated. This condition will be relaxed in the following sections, when more general mappings will be considered. For the rest of this thesis, any extended array manifold surface satisfying (2.52) will be called conformal extended array manifold surface, as it is produced by conformal complex mappings acting on the spatial manifold surfaces.

The tangent plane at a point \( \mathbf{h}(p, q) \) on the manifold surface is a plane surface spanned by the two tangent vectors \( \dot{\mathbf{h}}_p \) and \( \dot{\mathbf{h}}_q \). Therefore, if \( T_H \triangleq \left[ \dot{\mathbf{h}}_p, \dot{\mathbf{h}}_q \right] \), is a basis for the tangent plane of the extended array manifold surface and \( T \triangleq \left[ \dot{\mathbf{a}}_p, \dot{\mathbf{a}}_q \right] \) is a basis for the spatial array manifold surface, then these two bases are related as follows.

\[
T_H = \left[ \dot{\mathbf{h}}_p, \dot{\mathbf{h}}_q \right] = \left[ \mathbf{A} \dot{\mathbf{a}}_p, \mathbf{A} \dot{\mathbf{a}}_q \right] = \mathbf{A} T
\]

From Equation (2.53) one can deduce that the tangent plane of the extended array manifold surface at a specific point \( \mathbf{h}(p, q) \) will be rotated with regards to the tangent plane of the spatial array manifold surface at the corresponding point \( \mathbf{a}(p, q) \). This result was expected since the effect of the linear mapping \( T \) results in a rotation of the whole spatial manifold surface. Note, however, that this is not a trivial rotation such as, for example the rotation of a circle in a 3-dimensional real space, but it can be viewed as the rotation of a circle formerly lying on a
2.4 Extended Array Manifold Surfaces

Furthermore, based on Equation (2.53) the relationship connecting the two manifold metrics $G$ and $G_H$ can be derived.

$$G_H = \text{Re}\left\{ (T_H)^H \ T_H \right\}$$
$$= \text{Re}\left\{ (A \ T)^H \ A \ T \right\}$$
$$= \text{Re}\left\{ T^H \ (\sigma_A^2 I_N) \ T \right\}$$
$$= \sigma_A^2 \text{Re}\left\{ T^H T \right\}$$
$$= \sigma_A^2 G$$ \hspace{1cm} (2.54)

The elements of the manifold matrix are used to measure trajectories on the manifold surface via the concept of the first fundamental form $I$

$$I \triangleq dp^T G dp$$ \hspace{1cm} (2.55)

where

$$p(s) = [p(s), q(s)]^T$$

and

$$dp = [dp, dq]^T$$

Thus, Equations (2.54) and (2.55) imply that for the same infinitesimal increment $dp, dq$ of the directional parameters, the distance between $h(p, q)$ and $h(p + dp, q + dq)$ will be $\sigma_A$ times the distance between the corresponding spatial array manifold vectors. Moreover, the area with sides $dp, dq$ is mapped via the extended array manifold vector $h(p, q)$ onto an infinitesimal surface of an area $\sigma_A^2$ larger than the area of the surface which the spatial manifold vector produces. This is consistent with the result of the previous section that the new manifold is on a sphere with a radius $\sigma_A$ times larger than that of the spatial array manifold.

The Christoffel Matrices of the first kind $\Gamma_{1p,H}$ and $\Gamma_{1q,H}$ of the extended manifold surfaces are related to the corresponding matrices $\Gamma_{1p}$ and $\Gamma_{1q}$ of the
spatial manifold surfaces as follows

\[
\mathbf{\Gamma}_{1\zeta,\mathcal{H}} = \text{Re}\left\{ \mathbf{T}_\mathcal{H}^H \mathbf{T}_\zeta \right\} \\
= \text{Re}\left\{ (\mathbf{A}^T)^H \mathbf{A} \mathbf{T}_\zeta \right\} \\
= \sigma^2_A \text{Re}\left\{ \mathbf{T}_\mathcal{H}^H \mathbf{T}_\zeta \right\} \\
= \sigma^2_A \Gamma_{1\zeta}
\]

(2.56)

with \( \zeta = p, q \), while the Christoffel matrices of the second kind are identical.

\[
\mathbf{\Gamma}_{2\zeta,\mathcal{H}} = (\mathbf{G}_\mathcal{H})^{-1} \mathbf{\Gamma}_{1\zeta,\mathcal{H}} \\
= \frac{1}{\sigma^2_A} \mathbf{G}^{-1} \sigma^2_A \Gamma_{1\zeta} \\
= \mathbf{\Gamma}_{2\zeta}
\]

(2.57)

with \( \zeta = p, q \).

The Christoffel matrices are used to calculate the infinitesimal variance of the tangent plane as the point \( \mathbf{h}(p, q) \) moves on the manifold surface by \( d\mathbf{h} = \mathbf{\dot{h}}_p dp + \mathbf{\dot{h}}_q dq \), that is

\[
d\mathbf{T}_\mathcal{H} = \mathbf{T}_\mathcal{H} (\mathbf{\Gamma}_{2p,\mathcal{H}} dp + \mathbf{\Gamma}_{2q,\mathcal{H}} dq) \\
= \mathbf{A}^T (\mathbf{\Gamma}_{2p} dp + \mathbf{\Gamma}_{2q} dq) \\
= \mathbf{A}^T d\mathbf{T}
\]

However, since the shape of the actual surface has not changed, the only difference \( d\mathbf{T}_\mathcal{H} \) in the change of the vectors spanning the tangent plane in the case of the extended array manifold surface (compared to \( d\mathbf{T} \) of the spatial array manifold surface) is due to the original transformation of the tangent plane, which is described in Equation (2.53).

The final geometric property of surfaces, which will be examined is that of the Gaussian curvature. The sign of the Gaussian curvature of the array manifold surface at a point \( \mathbf{h}(p, q) \) provides an indication of the local shape of the surface at
the neighbourhood of this point. Since the shape of the spatial manifold surface has already been studied in the literature, it would be convenient to be able to link the shape of the extended array manifold surfaces to the corresponding spatial ones. The following theorem, Theorem 2, provides this link.

**Theorem 2.** Let $\mathcal{M} \triangleq \{ \mathbf{a}(p, q) \in C^n, \forall (p, q) : (p, q) \in \Omega_{p,q} \}$ be the spatial array manifold surface of an array of $N$ elements, the shape of which is described, locally at a point $\mathbf{a}(p, q)$, by the Gaussian curvature $K_g(p, q)$. Let

$$\mathcal{M}_H \triangleq \{ \mathbf{h}(p, q) \in C^q, \forall (p, q) : (p, q) \in \Omega_{p,q} \}$$

be an extended array manifold surface, related to $\mathcal{M}$ via the linear mapping $T : C^n \rightarrow C^q$. That is, $\mathbf{A} \in C^{q \times n}$ is the matrix representing the mapping $T$,

$$\mathbf{h}(p) = \mathbf{A}(p, q) \mathbf{a}(p, q)$$

If

$$\frac{\partial \mathbf{A}(p, q)}{\partial p} = \frac{\partial \mathbf{A}(p, q)}{\partial q} = \mathbf{0}$$

and

$$\mathbf{A}^H \mathbf{A} = \sigma_A^2 \mathbf{I}_N$$

then

1. The Gaussian curvature $K_{g,H}$ of the extended manifold surface $\mathcal{M}_H$ can be computed as

$$K_{g,H}(p, q) = \frac{1}{\sigma_A^2} K_g(p, q) \quad (2.58)$$

2. The shape of $\mathcal{M}_H$ at the neighbourhood of a point $\mathbf{h}(p_o, q_o)$ has locally the same shape as the spatial array manifold surface $\mathcal{M}$ has at a neighbourhood around $\mathbf{a}(p, q)$

**Proof.** The local shape of a surface at the neighbourhood of a point $\mathbf{h}(p_o, q_o)$ can be determined by the sign of the Gaussian Curvature at that point. Therefore, it is sufficient to prove Equation (2.58). For the rest of this proof, the dependence of all magnitudes on the parameters $p$ and $q$ will be dropped for notational convenience.

An expression for the Gaussian curvature of the surface $\mathcal{M}_H$, based on the
intrinsic geometry of the surface, is [63]

\[ K_{g,H}(p,q) = -\frac{1}{\sqrt{\det \{G_H\}}} \left( \frac{d}{dp} \left( \frac{\sqrt{\det \{G_H\}}}{g_{pp,H}} \Gamma^q_{pq,H} \right) \right) - \frac{d}{dq} \left( \frac{\sqrt{\det \{G_H\}}}{g_{pp,H}} \Gamma^q_{pp,H} \right) \]  

(2.59)

where

\[ g_{pp,H} \triangleq h_p^H \dot{h}_p = \sigma_A^2 \dot{\Delta}_p \dot{\Delta}_p = \sigma^2 g_{pp} \]

and

\[ \Gamma_{2\zeta,H} = \begin{bmatrix} \Gamma^\zeta_{pp,H} & \Gamma^\zeta_{pq,H} \\ \Gamma^\zeta_{qp,H} & \Gamma^\zeta_{qq,H} \end{bmatrix} = \begin{bmatrix} \Gamma^\zeta_{pp} & \Gamma^\zeta_{pp} \\ \Gamma^\zeta_{qp} & \Gamma^\zeta_{qq} \end{bmatrix}, \quad \zeta = p, q \]

Thus, by substituting into Equation (2.59)

\[ K_{g,H} = -\frac{1}{\sigma_A^2 \sqrt{\det \{G\}}} \left( \frac{d}{dp} \left( \frac{\sqrt{\det \{G\}}}{\sigma_A^2 g_{pp}} \Gamma^q_{pq} \right) \right) - \frac{d}{dq} \left( \frac{\sqrt{\det \{G\}}}{\sigma_A^2 g_{pp}} \Gamma^q_{pp} \right) \]

\[ = \frac{1}{\sigma_A^2} \left( -\frac{1}{\sqrt{\det \{G\}}} \left( \frac{d}{dp} \left( \frac{\sqrt{\det \{G\}}}{g_{pp}} \Gamma^q_{pq} \right) \right) - \frac{d}{dq} \left( \frac{\sqrt{\det \{G\}}}{g_{pp}} \Gamma^q_{pp} \right) \right) \]

\[ = \frac{1}{\sigma_A^2} K_g \]  

(2.60)

This relationship between the two Gaussian curvatures was to be expected, since it reflects the fact that the extended array manifold surface \( \mathcal{M}_H \) lies on a hypersphere of radius \( \sigma_A \) times larger than that on which the spatial array manifold surface \( \mathcal{M} \) lies.

Finally, let us assume that there exists a spatial array manifold curve \( \mathcal{A} \) on the spatial manifold surface \( \mathcal{M} \) and the corresponding, via a linear mapping \( T \) which satisfies the conditions of Section 2.2, extended array manifold curve \( \mathcal{A}_H \) on \( \mathcal{M}_H \).
The properties of the extended manifold curves have been extensively studied in Section 2.3. However, for curves lying on surfaces, another geometric property of interest is the geodesic curvature $\kappa_g$ of the curve. The geodesic curvature is defined as the component of the principal curvature $\kappa_1$ along the tangent plane to the surface at every point of the curve. If the geodesic curvature of a curve $A$ on a surface $M$ is equal to zero throughout its length, then this curve is called geodesic and it defines the path of minimum length on $M$, which connects the two end points of $A$. The concept of geodicity of a curve is extremely important, as it assesses the similarity of of the curve to a straight line lying on a plane. Therefore, it is interesting examine how the geodesic curvature $\kappa_{g,H}(s)$ of the extended manifold curve $A_H$ is connected to the geodesic curvature $\kappa_g(s)$ of the spatial manifold curve $A$.

Let us consider the formula for the geodesic curvature given in [31].

$$\kappa_g(s) = \sqrt{\det \{G(s)\}} \frac{\partial p}{\partial s}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \left( \frac{\partial^2 p}{\partial s^2} + \left( \Gamma_{2p} \frac{\partial p}{\partial s} + \Gamma_{2q} \frac{\partial q}{\partial s} \right) \frac{\partial p}{\partial s} \right)$$

(2.61)

where

$$p(s) = [p(s), q(s)]^T$$

For an extended array manifold curve, Equation (2.61) can be reformulated as

$$\kappa_{g,H}(s) = \sqrt{\det \{G_H(s)\}} \frac{\partial p_H}{\partial s}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \left( \frac{\partial^2 p_H}{\partial s^2} + \left( \Gamma_{2p,H} \frac{\partial p_H}{\partial s} + \Gamma_{2q,H} \frac{\partial q_H}{\partial s} \right) \frac{\partial p_H}{\partial s} \right)$$

(2.62)

Since the objective is to compare two manifold curves which have, in general, different lengths but the same domain for the bearing parameters $p$ and $q$, it is more convenient to consider Equations (2.61) and (2.62) a functions of $p$ and $q$ and not $s$. In other words, the point $a(p_o, q_o)$ on $A$ (and of course on $M$) is
mapped via $T$ onto the point $h(p_o, q_o)$ on $\mathcal{A}_H$ (and $\mathcal{M}_H$). However, $\mathbf{a}(s_o)$ is not mapped onto $h(s_o)$, since as it was shown in Section 2.3 the total manifold curve length of $\mathcal{A}$ is different than that of $\mathcal{A}_H$.

For the extended array manifold curve of Equation (2.62)

$$\frac{\partial \mathbf{p}_H}{\partial s} = \left( \frac{\partial s}{\partial \mathbf{p}_H} \right)^{-1} = \left[ \frac{1}{\|h_p\|}, \frac{1}{\|h_q\|} \right]^T = \frac{1}{\sigma_h} \frac{\partial \mathbf{p}}{\partial s}$$ (2.63)

and

$$\frac{\partial^2 \mathbf{p}_H}{\partial s^2} = \frac{\partial}{\partial s} \left( \frac{\partial \mathbf{p}_H}{\partial s} \right) = \frac{1}{\sigma_h} \frac{\partial^2 \mathbf{p}}{\partial s^2}$$ (2.64)

Therefore

$$\kappa_{g,H}(p, q) = \sigma_h^2 \sqrt{\det \{G(s)\}} \frac{1}{\sigma_h} \frac{\partial \mathbf{p}}{\partial s}^T \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \frac{1}{\sigma_h} \left( \frac{\partial^2 \mathbf{p}}{\partial s^2} + \left( \Gamma_{2p} \frac{\partial \mathbf{p}_H}{\partial s} + \Gamma_{2q} \frac{\partial q}{\partial s} \right) \frac{\partial \mathbf{p}}{\partial s} \right)$$

$$= \kappa_g(p, q)$$ (2.65)

Note, of course that since the geodesic curvature is a property associated with the manifold curve, it depends on one parameter only which is consistent with the notational convention $\kappa_g(p, q)$ since there is a dependency between $p$ and $q$. The main conclusion drawn from Equation (2.65) is that geodesic spatial manifold curves are mapped to geodesic extended manifold curves.

### 2.4.2 “Non-conformal” extended manifold surfaces

In the previous section, the differential geometric properties of the extended array manifold surfaces arising from conformal complex mappings (i.e. $\mathbf{A}$ independent of $p, q$) of the spatial array manifold surfaces have been examined. The attention is now turned to extended array manifold surfaces which are the product of non-conformal mappings. Namely, conditions (2.52) are relaxed and now the matrix $\mathbf{A} = \mathbf{A}(p, q)$ is a function of both $p$ and $q$. These extended array manifold
surfaces will be referred to for the rest of this thesis as *non-conformal* extended array manifold surfaces. It is expected that the non-conformality of the complex mapping will deform the spatial array manifold surface and the relatively simple connection between the geometric properties of the two surfaces will not be present any more.

The investigation will commence with the tangent plane at some point \( h(p, q) \) of the extended manifold surface. Similarly to Equation (2.53), which is true for conformal mappings, \( T_H \) for non-conformal mappings can be expressed as follows.

\[
T_H = \left[ \dot{h}_p, \dot{h}_q \right] = \left[ \dot{A}_p \alpha, \dot{A}_q \alpha + \dot{A}_\alpha \right] = A_T + \dot{A}_{pq} (I_2 \otimes \alpha) \tag{2.66}
\]

where

\[
\dot{A}_{pq} \triangleq \left[ \dot{A}_p, \dot{A}_q \right] = \left[ \frac{\partial A}{\partial p}, \frac{\partial A}{\partial q} \right] \tag{2.67}
\]

Note that now the basis of the tangent plane of the extended manifold surface is comprised of two terms. The first term is identical to Equation (2.53), while the second one represents the deformation of the surface because of the dependence of the complex mapping on the bearing parameters \( p \) and \( q \).

Based on Equation (2.66) the relationship connecting the two manifold metrics \( G \) and \( G_H \) can be derived.

\[
G_H = \text{Re} \left\{ (T_H)^H T_H \right\} = \text{Re} \left\{ (A_T + \dot{A}_{pq} I_2 \otimes \alpha)^H \left( A_T + \dot{A}_{pq} I_2 \otimes \alpha \right) \right\} = \text{Re} \left\{ (A_T)^H A_T + \dot{A}_{pq} I_2 \otimes \alpha \dot{A}_{pq} I_2 \otimes \alpha + 2 \dot{A}_{pq} I_2 \otimes \alpha A_T \right\} = \text{Re} \left\{ (A_T)^H A_T + (I_2 \otimes \alpha)^H (G_\alpha \otimes I_N) (I_2 \otimes \alpha) \right\} = \sigma_\alpha^2 G + NG_\alpha \tag{2.68}
\]
where, in similarity to the definitions of $\mathcal{G}$ and $\mathcal{G}_H$, $\mathcal{G}_z$ is defined as

\[
\mathcal{G}_z \triangleq \begin{bmatrix}
\dot{z}_p^H \dot{z}_p & \text{Re} \left\{ \dot{z}_p^H \dot{z}_q \right\} \\
\text{Re} \left\{ \dot{z}_q^H \dot{z}_p \right\} & \dot{z}_q^H \dot{z}_q
\end{bmatrix}
\] (2.69)

Let us compare Equation (2.68) with Equation (2.54). For an non-conformal extended array manifold surface there are two terms contributing to the manifold metric $\mathcal{G}_H$. The first term is the same as the one in Equation (2.54) and expresses the change of dimensionality of the embedding complex space. The second term $NG_z$, however, is related to the dependence of the mapping on the directional parameters and can be visualised as the manifold metric $\mathcal{G}_z$ of a second surface, whose parametric equation is given by

\[
\mathcal{M}_z = \left\{ \tilde{z} (p, q) \in \mathbb{C}^Q : (p, q) \in \Omega_{p,q} \right\}
\] (2.70)

Note, however, that the manifold surfaces are embedded in complex spaces of different dimensions, that is $\mathcal{M} \subset \mathbb{C}^N$, $\mathcal{M}_H \subset \mathbb{C}^Q$ and $\mathcal{M}_z \subset \mathbb{C}^Q$. Therefore, another interpretation of the extended array manifold surface is that the spatial array manifold vector has acted on $\mathcal{M}_z$ to produce $\mathcal{M}_H$. However, the properties of this new mapping are not entirely dual to those of $T$ and this can be deduced from the first term on the right hand side of Equation (2.66), where the term $\hat{\hat{a}}_{pq} = [I_N \otimes \dot{z}_p, I_N \otimes \dot{z}_q]$ appears instead of $[\dot{z}_p, \dot{z}_q]$ which would be the dual of $T$. This, nevertheless, is simply related to the way one chooses to view the output of the Rx-LPU and stems from the structure of the output of the Tapped Delay Line used in the literature so far.

Let us proceed next to the Christoffel matrices of the first and second kind. Following a detailed analysis which can be found in Appendix B

\[
\Gamma_{1,\zeta,H} = \text{Re} \left\{ T_H^H \hat{T}_{\zeta,H} \right\} = \text{Re} \left\{ T_H^H \tilde{\tilde{z}} \hat{\partial}_{\tilde{\zeta}} T + \dot{\tilde{z}}^H \tilde{\tilde{z}} \tilde{\tilde{G}} + \hat{T}_{\zeta}^H \hat{a} \dot{\tilde{z}}^H T \tilde{z} \right\} + N \Gamma_{1,\zeta,z} + \sigma^2_\mathcal{A} \Gamma_{1,\zeta}
\] (2.71)
where

\[ T(z) \triangleq \begin{bmatrix} \dot{z}_p, \dot{z}_q \end{bmatrix} \tag{2.72} \]

is the tangent matrix of \( M_z \).

\[ \dot{T}_{\zeta,z} \triangleq \begin{bmatrix} \ddot{z}_{p\zeta}, \ddot{z}_{q\zeta} \end{bmatrix} \tag{2.73} \]

\[ \Gamma_{1\zeta,z} \triangleq \text{Re} \left\{ \dot{T}_{\zeta,z}^H T(z) \right\} \tag{2.74} \]

is the Christoffel matrix of the first kind for \( M_z \) and, finally

\[ \tilde{G} \triangleq \begin{bmatrix} \dot{a}_p^H \dot{a}_p, \dot{a}_q^H \dot{a}_q \\ \dot{a}_p^H \dot{a}_q, \dot{a}_q^H \dot{a}_p \end{bmatrix} \tag{2.75} \]

Again, the pattern which appeared initially in the study of the extended manifold curves and next in the tangent matrix \( T_H \) and the manifold metric \( G_H \) is present. On the right hand side of Equation (2.71) there exist two terms, which are properly scaled versions of the Christoffel matrix of the first kind \( \Gamma_{1\zeta,H} \) of the extended array manifold surface \( M_H \) and the Christoffel matrix of the first kind \( \Gamma_{1\zeta,z} \) of \( M_z \). The remaining terms,

\[ \text{Re} \left\{ \dot{T}_{\zeta,z}^H \dot{a}_\zeta^H \dot{T} + \dot{a}_\zeta^H \tilde{G} + \dot{T}_{\zeta,z}^H \tilde{a}_\zeta^H T(z) \right\} \]

are due to the fact that the reference point of the array system is taken to be the array centroid. This guarantees that

\[ \frac{1}{N} \left[ \begin{array}{c} l_x, l_y, l_z \end{array} \right] = \underline{0}_T \]

and therefore \( \dot{a}_\zeta^H \underline{a} = 0 \), \( \zeta = p, q \) but does not guarantee that the tangent vectors \( \dot{z}_\zeta(p,q) \), \( \zeta = p, q \) at every point \( \underline{z}(p,q) \) of the manifold surface \( M_{\underline{z}} \) will be normal to the surface.

Finally, the Christoffel matrix of the second kind for the extended manifold surface is related to the respective matrix of the spatial array manifold surface.
as follows

\[ \Gamma_{2\zeta,\mathcal{H}} = (\mathcal{G}_{\mathcal{H}})^{-1}\Gamma_{1\zeta,\mathcal{H}} \]

\[ = \frac{\sigma^2 |\mathcal{G}| \mathcal{G}^{-1} + N |\mathcal{G}_{\mathcal{Z}}| \mathcal{G}_{\mathcal{Z}}^{-1}}{N^2 |\mathcal{G}_{\mathcal{Z}}| + \sigma^4 |\mathcal{G}| + N \sigma^2 \mathcal{A} \mathcal{G} \mathcal{G}^{-1}} \]

\[ \cdot \left( \text{Re} \left\{ T^H_{\mathcal{Z}} \mathcal{A}^H \mathcal{T} + \mathcal{Z}^H \mathcal{G} + \mathcal{T}^H \mathcal{A} \mathcal{Z}^H \mathcal{T} \right\} + N \Gamma_{\mathcal{Z},1\zeta} + \sigma^2 \mathcal{A} \Gamma_{1\zeta} \right) \]

\[ (2.76) \]

### 2.5 Summary

In the Chapter the concept of the extended array manifold has been introduced and its functional dependence on the spatial array manifold has been investigated. In detail, the geometric properties of extended array manifold curves and extended manifold surfaces have been studied as a function of the corresponding properties of the spatial array manifolds. Throughout the Chapter, a distinction has appeared between extended array manifolds (conformal extended manifolds) which retain, locally, the shape of the spatial array manifolds and non-conformal extended manifolds which appear from non-conformal complex mappings of the spatial array manifolds.
Two important measures for the analysis of the performance of an array system are

- the detection threshold and
- the resolution threshold

introduced in [34].

The detection capability of an array system is defined as its ability to detect two closely located sources for a given Signal-to-Noise ratio (SNR) of each individual signal. The resolution capability is related to the ability of the array to actually resolve the two directions of the incoming signals, for a given SNR. If an infinite observation interval were available at the array, the sources would be detected and resolved irrespectively of how small the angular separation between them was. It is clear, therefore, that the detection/resolution issue exists because of finite observation interval. It is, thus, preferable to express the detection/resolution thresholds as the minimum number of snapshots, \( L_{\text{det}} \) and \( L_{\text{res}} \) respectively, that are required in order to detect/resolve two sources that are separated by some small angle \( \Delta p \), for a given SNR.

Up to now, the detection and resolution thresholds have been estimated only for the case of linear arrays [34] and for planar arrays [31]. Furthermore, the
analysis in these papers has focused only on the case of spatial array manifolds and how the geometry of the spatial array manifolds poses a limit on the detection and resolution capabilities of the array systems.

A similar investigation has been performed in [10] where the authors study the detection capabilities of array systems based on the notion of the effective rank of the channel matrix. Note that in this paper, the authors use the term “resolution” for what in the thesis is termed as detection. Moreover, the authors do not investigate the resolution capabilities of the array systems.

In [2], the authors study the Cramer Rao Bound for the accuracy of DOA estimation for and propose a method for designing array geometries which achieve uniform performance. Some of the results of this paper seem to duplicate those of [34], however they do not derive bounds for the detection and resolution capabilities of random array geometries.

In the case of a fully 3-dimensional environment, array geometries where sensors do not lie on the same horizontal plane may be formed and, furthermore, these arrays have to be able to resolve signals from sources anywhere in the 3-dimensional space. Consequently, the theoretical detection and resolution thresholds need to be extended, in order to encompass the more general 3-dimensional case, as well as the case where extended array manifolds are used for the modelling of the array system.

Manifolds of 3-dimensional array geometries are the most general case of array manifolds. These manifolds are functions of both azimuth and elevation and, therefore, represent a surface embedded in a multidimensional complex space, as has already been shown in Chapter 2. However, manifold surfaces for 3-dimensional array geometries are difficult to be studied.

Finally, it is important to note that the concepts of the detection and resolution thresholds of array systems, as they have been introduced in [34], are due to the geometry of the array and they constitute the absolute lowest performance limits of the array system. The various algorithms which are used to actually de-
tect the signals and resolve their DOAs may or may not achieve this theoretical lower bound.

### 3.1 Array Bounds for 3D Array Geometries

The aforementioned bounds have so far been estimated for linear array geometries and for planar arrays, in the case of sources of constant elevation, constant azimuth of constant $\alpha$ or $\beta$ cone angles. For planar arrays via the concept of Equivalent Linear arrays (ELA). Let us assume that two co-channel sources, with bearings $(p_1, q_o)$ and $(p_2, q_o)$ are impinging on an array, the geometry of which is given by the coordinates matrix $[r_x, r_y, r_z]$. The condition that the elevation angle is fixed, i.e. $q = q_o$ defines a curve

$$A_{p|q_o} \triangleq \{a(p, q_o), \forall p : p \in \Omega_p\} \quad (3.1)$$

which lies on the spatial array manifold surface $\mathcal{M}$ of Equation (1.1).

The approach presented in [34] is based on a circular approximation of the array manifold curve $A_{p|q_o}$ at the neighbourhood of the point $(\tilde{p} = \frac{p_1 + p_2}{2}, q_o)$. Based on this approximation the detection and resolution thresholds are found to be

$$\Delta p_{\text{det}} = \frac{1}{\sqrt{2s(\tilde{p})}} \left( \frac{1}{\sqrt{\text{SNR}_1 \cdot L}} + \frac{1}{\sqrt{\text{SNR}_2 \cdot L}} \right) \quad (3.2)$$

and

$$\Delta p_{\text{res}} = \frac{1}{s(\tilde{p})} \sqrt{\frac{2}{\hat{\kappa}_1^2 - \frac{1}{N}}} \left( \frac{1}{\sqrt{\text{SNR}_1 \cdot L}} + \frac{1}{\sqrt{\text{SNR}_2 \cdot L}} \right) \quad (3.3)$$

where $\text{SNR}_i$, $i = 1, 2$ is the Signal-to-Noise ratio of the $i$-th source, $p_1$ is the bearing of the first source, $p_2 = p_1 + \Delta p$ is the bearing of the second source, $L$ is the number of the snapshots available for the detection/estimation of $p_1, p_2$ and

$$\hat{\kappa}_1 = \kappa_1 \sin(\zeta) \quad (3.4)$$

where $\kappa_1$ is the principal curvature of the spatial array manifold curve at $\tilde{p}$ and $\zeta$ is the angle between the first two coordinate vectors (Frenet Frame vectors
3.1 Array Bounds for 3D Array Geometries

which span the principal tangent plane on which the circular approximation lies) $u_1$ and $u_2$ at that point [21].

However, for random 3-dimensional geometries and sources impinging from any possible bearing of the 3D space, no ELA can be found (as will be proven in Chapter 4), which produces a spatial array manifold curve that includes both points of the manifold surface. Therefore, in this chapter another approach has been adopted. The $\alpha, \beta$ parametrisation will be used in order to get a curve, which actually includes both points of interest on the manifold surface. After this curve has been identified, an approximation of this curve at the neighbourhood of the points of interest will provide the required geometric parameters of interest in order to express the required theoretical bounds.

3.1.1 Defining the $(\alpha, \beta)$ Parametrization

Although the use of polar angles $\theta$ and $\phi$ is the most common in order to parameterize the array manifold, in certain cases, the analysis of the latter is facilitated by the use of an alternative parametrisation called the cone-angle parametrisation, which was introduced in [23]. With reference to Figure 3.1 (see [31]), the basic transformation formulae are

\[
\begin{align*}
\cos \alpha &= \cos (\theta - \Theta_o) \cos (\phi) \\
\cos \beta &= \sin (\theta - \Theta_o) \cos (\phi)
\end{align*}
\] (3.5)

where $\Theta_o$ is the angle by which the $x - y$ frame of the axes has been rotated counter-clockwise to form a new set of axes $\hat{x}, \hat{y}$ for the $x - y$ plane. The cone angle $\alpha$ is the angle between the wavenumber vector $\hat{k}$ with the positive side of the $\hat{x}$ axis, while the cone angle $\beta$ is the angle between the same wavenumber vector and the positive side of the $\hat{y}$ axis.

3.1.2 Using the $(\alpha, \beta)$ Parametrization

Let the positions of the array elements be given by a matrix $r \triangleq [r_x, r_y, r_z]$ and let two incoming signals with directions of arrival $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ respectively.
Note that now both the azimuth and the elevation bearing parameters of the incoming sources are different. In order to find an $\alpha$-curve$^1$ which passes through both these points of the manifold surface, the parameter $\beta$ is considered to be constant and equal to $\beta_o$. Based on Equation (3.5)

\[
\begin{align*}
\cos \beta_o &= \sin (\theta_1 - \Theta_o) \cos \phi_1 \\
\cos \beta_o &= \sin (\theta_2 - \Theta_o) \cos \phi_2 \\
\Theta_o &= \arctan \left( \frac{\sin \theta_1 \cos \phi_1 - \sin \theta_2 \cos \phi_2}{\cos \theta_1 \cos \phi_1 - \cos \theta_2 \cos \phi_2} \right)
\end{align*}
\]  

(3.6)

Based on this transformation the spatial array manifold vector can be expressed as

\[
\hat{a}(\alpha) = \exp \left( -j \vec{r} \cdot \hat{k}(\alpha) \right)
\]

(3.7)

$^1$An $\alpha$-curve is defined as $A_{\alpha|\beta_o} \triangleq \{ \hat{a}(\alpha, \beta) : \alpha \in \Omega_\alpha, \beta = \beta_o \}$
where \( \vec{r} = \begin{bmatrix} \vec{r}(\Theta_o) \ , \ \vec{r}(\Theta_o + 90^\circ) \ , \ \vec{r}_z \end{bmatrix} \)
\[
\vec{r}(\Theta_o) \triangleq \vec{r}_x \cos(\Theta_o) + \vec{r}_y \sin(\Theta_o)
\]
and
\[
k(\alpha) = \pi \begin{bmatrix} \cos (\alpha) \ , \ \cos (\beta_o) \ , \ \sqrt{1 - \cos^2 (\alpha) - \cos^2 (\beta_o)} \end{bmatrix}^T
\]

Now that the desired \( \alpha \)-curve has been identified, the rate of change of the arc-length \( \dot{s} \)\( \alpha \)\( (\alpha) \triangleq \frac{ds(\alpha)}{d\alpha} \), the principal curvature \( \kappa_1 \) of the spatial array manifold curve and the inclination angle \( \zeta \) have to be calculated. Using these values for the parameters \( \Theta_o \) and \( \beta_o \) of the cone-angle parametrisation, it is possible to prove (see Appendix C) that the rate of change of the arc-length \( \dot{s} \alpha \), the principal curvature \( \kappa_1 \) of the \( \alpha \)-curve at the point of interest (which is the curvature of the circular approximation of the \( \alpha \)-curve at the same point) and the inclination angle \( \zeta \) are given as follows.

\[
\dot{s} \alpha (\alpha) \triangleq \| \vec{w}(\alpha) \|
\]
\[
\kappa_1 (\alpha) = \left\| \frac{w^2(\alpha)}{\|w^2(\alpha)\|^2} + \frac{j}{\pi \sin \alpha \|w(\alpha)\|} (G(\alpha) \vec{r}(\Theta_o) - H(\alpha) \vec{r}_z) \right\|
\]
where
\[
G(\alpha) = \sin \alpha \frac{\left\| \vec{r}_x^2 \cos \alpha \right\|}{g(\alpha)} - \frac{\vec{r}(\Theta_o)^T \vec{r}_z}{\left\| \vec{r}(\Theta_o) - \frac{\cos \alpha}{g(\alpha)} \vec{r}_z \right\|^3}
\]
\[
H(\alpha) = - \frac{\sin \alpha}{g(\alpha)} \left( \frac{1 + \cos^2(\alpha)}{g(\alpha)} \right) + \frac{\cos \alpha}{g(\alpha)} G(\alpha)
\]
\[
\vec{w}(\alpha) = \vec{r}(\Theta_o) - \frac{\cos \alpha}{\sqrt{1 - \cos^2(\alpha) - \cos^2(\beta_o)}} \vec{r}_z
\]

Having calculated \( \kappa_1 (\alpha) \), it is easy to calculate the angle between the first two coordinate vectors \( \vec{u}_1(\alpha) \) and \( \vec{u}_2(\alpha) \)
\[
\sin \zeta (\alpha) = \sqrt{1 - |\vec{u}_1 H(\alpha) \cdot \vec{u}_2(\alpha)|^2}
\]
where
\[ u_1(\alpha) = \frac{\dot{a}(\alpha)}{\|\dot{a}(\alpha)\|} = j \frac{w(\alpha)}{\|w(\alpha)\|} \odot a(\alpha) \] (3.14)
and
\[ u_2(\alpha) = \frac{1}{\kappa_1} \frac{\dot{u}_1(\alpha)}{\|\dot{a}(\alpha)\|} \] (3.15)

It should be noted that \( \sin \zeta \) in a real orthogonal system of coordinates should be equal to 0. However, in this case the imaginary part of the inner product is not 0, which implies that the two coordinate vectors \( u_1 \) and \( u_2 \) are not orthogonal. This is the reason why the curvature of the circular approximation is not the same as the magnitude of \( \frac{\partial u_1}{\partial s} \).

### 3.1.3 Resolution and Detection Thresholds for the Spatial Manifold

In the previous section, the geometric parameters \( \kappa_1, \dot{s}_\alpha \) and \( \sin \zeta \) have been derived for any 3-dimensional array geometry and, thus, the detection and resolution bounds, given by Equations (3.2) and (3.3) can be evaluated. In this section, two representative examples of 3-dimensional geometries will be studied:

1. A 3-dimensional grid geometry.

2. A 3-dimensional array of random, known geometry.

#### 3-dimensional “grid” array geometry

The array under consideration is pictured in Figure 3.2. This array is a representative example of the class of so-called grid arrays, the geometry of which satisfies the constraint
\[ [r_x, r_y, r_z]^T [L_x, L_y, L_z] = cI_3 \] (3.16)

This array operates in the presence of two sources with directions of arrival \((\theta_1, \phi_1)\), \((\theta_1 + \Delta \theta, \phi_1 + \Delta \phi)\). The two sources are considered equipowered and the system is corrupted by AWGN. The required number of snapshots \( L \) (observation interval) in Equations (3.2), (3.3) will be evaluated with \((\theta_1, \phi_1)\) taking values
in the upper hemisphere of the 3-dimensional space. It is important to point out that the case where \( \phi = 0 \) has been excluded because the \((\alpha, \beta)\) parametrization cannot be used at that point. If \( \phi_1 = \phi_2 = 0 \), the bounds for the corresponding \( \theta \)-curve can be derived using the methodology presented in [31].

In Figures 3.3 - 3.4 the required number of snapshots for the detection of the two sources have been plotted. It is interesting to notice that for a given elevation angle \( \phi = \phi_o \) the detection threshold is constant and independent of the azimuth angle \( \theta \). This is a property of 3-dimensional grid arrays due to the independence of the rate of change of the arc length \( \frac{ds}{d\alpha} \) of the azimuth. As it will be shown later on, this is not the case for all 3-dimensional arrays in general.

Furthermore, it is interesting to comment on the behaviour of the detection threshold as \( \phi \) approached 90°. Although one would expect a similar behaviour as for the case when \( \phi = \theta = 0^\circ \), this is not the case, because for the same angular

* Figure 3.2: Array of 6 elements. The elements of the array are placed at the centre of each side of a cube with edge equal to 2 half-wavelengths.*
3.1 Array Bounds for 3D Array Geometries

separation between the two sources, the distance between the array manifold
vectors is much smaller in the former case than the latter.

Another point that should be noted is that the results presented in this analy-
isis refer to the minimum number of snapshots required for the detection/resolution
of two sources, based on the limit posed by the array geometry. However, there
are other factors which may necessitate a higher limit on the minimum number of
snapshots required. For example, the minimum number of snapshots cannot be
less than the number of the array elements, otherwise the matrix $R_{xx}$ will not be
of full rank and therefore, it will not represent accurately the whole observation
space. This is the reason why values of $L_{\text{det}}$ smaller than the number of antennas
$N$ can be seen in Figure 3.4. Therefore the required number of snapshots is given
by the following expression

$$L_{\text{required}} = \max\{L_{\text{det}}, N\}$$

where $L_{\text{det}}$ is the value calculated above and $N$ is the number of antennas.

Finally, in Figures 3.5 and 3.6, the required snapshots for resolving two sources
being separated by $0.5^\circ$ both in the azimuth and the elevation are given.
3.1 Array Bounds for 3D Array Geometries

Figure 3.3: Contour plot of the detection capability for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2

Figure 3.4: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2
Figure 3.5: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB of the array of Figure 3.2

Figure 3.6: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2
From figures 3.5 and 3.6 a number of observations can be made.

1. The maximum required number of snapshots $L_{\text{res}}$ corresponds to an elevation equal to 45°. This is an expected result due to the geometry of the array.

2. In addition, again due to the 4-fold symmetry of the cube, for a given elevation angle $\phi$, the same pattern is repeated 4 times.

3. One may observe that the whole pattern is somehow tilted downwards in the contour plot. This can be explained by the way this plot has been produced. It can be seen from Equation (3.3) that the various variables are evaluated at the point $\bar{p} = \frac{P_1 + P_2}{2}$, with the two points being on a line on the $(\theta, \phi)$ plane with positive inclination, since the second source is at bearing $(\theta_1 + \Delta\theta, \phi_1 + \Delta\phi)$. If the second bearing was at $(\theta_1 \pm \Delta\theta, \phi_1 \mp \Delta\phi)$ then the plot would be “tilted” upwards, since the two points on the $(\theta_1 \pm \Delta\theta, \phi_1 \mp \Delta\phi)$ plane, representing the two sources, would be on a line with negative inclination.

4. Another point, which is worth commenting on, is the behaviour of this plot for $\phi = 90^\circ$. Although for $\phi = 90^\circ$ all $\theta \in [0, 2\pi)$ correspond to the same point in the 3-dimensional space, one may observe a fluctuation in the value of the required resolution snapshots. The reason for this is that there are infinite curves passing through the point on the manifold surface for which $\phi = 90^\circ$. However, each of these curves has a different curvature $\kappa_1$. Thus, whenever a value of $\Theta_o$ is calculated (for every possible value of $\theta$) using Equations (3.5), a new $\alpha$-curve is selected, which is rotated on the manifold surface. All these $\alpha$-curves do not have the same curvature at the point under consideration and this is the reason why the value of $L_{\text{res}}$ fluctuates. A plot for $\kappa_1$ at $\phi = 90^\circ$ for the different $\alpha$-curves can be found in Appendix C.
Random 3-dimensional array

In Figure 3.7 the positions of the elements of a 3-dimensional array of random, but given, geometry are plotted. The coordinates of the array elements are given in Table 3.1. The detection and resolution thresholds for this array are given in Figures 3.8 - 3.11.

![Figure 3.7: Random array of 6 elements](image)

Table 3.1: Coordinates of the elements of the array of Figure 3.7

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.025</td>
<td>-0.261</td>
<td>-0.977</td>
</tr>
<tr>
<td>0.941</td>
<td>-0.133</td>
<td>0.486</td>
</tr>
<tr>
<td>-0.730</td>
<td>-0.141</td>
<td>-0.118</td>
</tr>
<tr>
<td>-0.140</td>
<td>0.669</td>
<td>0.856</td>
</tr>
<tr>
<td>0.780</td>
<td>0.006</td>
<td>-0.076</td>
</tr>
<tr>
<td>-0.826</td>
<td>-0.141</td>
<td>-0.171</td>
</tr>
</tbody>
</table>
3.1 Array Bounds for 3D Array Geometries

Figure 3.8: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB of the array of Figure 3.7

Figure 3.9: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.7
Figure 3.10: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20 dB of the array of Figure 3.7

Figure 3.11: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20 dB for the array of Figure 3.7
Based on these Figures, it can be observed that for a purely random 3-
dimensional array geometry, the detection capability of the array is not independent of the azimuth, in contrast to the 3-dimensional grid array used in the previous example. The detection and resolution thresholds for random geometries and sources of random bearings do not exhibit some definitive pattern (due to the lack of symmetry), as can be seen from Figures 3.8 - 3.11. However, it is obvious that the performance of an array system of random geometry can suffer significantly in certain directions in space, such as the direction $(114^\circ, 55^\circ)$ for the array of Figure 3.7. Therefore, the expressions presented in this chapter may be of value in the decision for the formation of dynamic arrays in environments where multiple nodes are available to serve as the elements of a wireless array system (e.g. in arrayed wireless sensor networks). The detection and resolution capabilities towards specific directions of interest could serve as decision criteria for the selection of certain nodes to serve as elements in a wireless array with desired operational characteristics.
3.2 Array Bounds Based on the Extended Array Manifolds

The analysis presented in Section 3.1 focused on the theoretical performance bounds of array systems with regards to their ability to detect and resolve sources, the bearings of which are arbitrarily close. Equivalently, these bounds reflect the ability of the array systems to detect and resolve the two spatial array manifold vectors associated with the sources under consideration. A natural extension of this work is in the case of systems that are modelled with the use of the extended array manifold vectors.

In the following investigation two separate cases will be considered, one involving extended array manifolds emerging as conformal mappings (see Section 2.4.1) of the spatial array manifold and one involving non-conformal mappings (see Section 2.4.2). The reason of this distinction is related to the deformation of the spatial array manifold surface in the case of non-conformal mappings. This deformation, as will be shown later on, drastically affects the detection and resolution bounds of the array systems. In addition, a new approach in the estimation of the detection bound will be presented. This approach will be shown to be compatible with the effects of the extensions of the spatial array manifold surface irrespectively of the nature of the extension of the spatial manifold surface and will be presented in Section 3.3. However, first, the modifications to the expressions of Section 3.1 will be presented, in order to deal with “conformal” and “non-conformal” extended array manifolds.

3.2.1 Array Bounds of “Conformal” Extended Manifolds

In general, the complex mappings producing the extended array manifolds are dependent on the bearing parameters $p$ and $q$ of the associated spatial manifold. However, the basic assumptions, on which the derivation of Equations (3.2) and (3.3) was based, hold whether $A$ is dependent on $p, q$ or not. The reason for this is that, as it has been demonstrated in Chapter 2, the complex mapping
results in a new manifold curve which again can be approximated locally with a circular arc located on the plane spanned by the first two coordinate vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Therefore, the corresponding formulae for the case of conformal extended array manifolds will be of the same format, with the exception that the geometric properties of interest will be those of the respective extended manifold curve.

According to the results of Chapter 2, the relationship between the required geometric properties of the extended and the associated the spatial manifold curve can be summarised as follows.

$$\dot{s}_{\mathcal{H}}(p) = \sigma_{\mathcal{H}} \dot{s}(p) \quad (3.18a)$$

$$\kappa_{1,\mathcal{H}} = \frac{1}{\sigma_{\mathcal{H}}} \kappa_1 \quad (3.18b)$$

$$\mathbf{u}_{1,\mathcal{H}}(p) = \frac{1}{\sigma_{\mathcal{H}}} \hat{A}_{\mathcal{H}} \mathbf{u}_1(p) \quad (3.18c)$$

$$\mathbf{u}_{2,\mathcal{H}}(p) = \frac{1}{\sigma_{\mathcal{H}}} \hat{A}_{\mathcal{H}} \mathbf{u}_2(p) \quad (3.18d)$$

$$\sin \zeta_{\mathcal{H}}(p) = \sqrt{1 - \left| \mathbf{u}_{2,\mathcal{H}}(p) \mathbf{u}_{1,\mathcal{H}}(p) \right|^2} = \sin \zeta(p) \quad (3.18e)$$

Thus, the detection and resolution thresholds for systems modelled using “conformal” extended array manifolds can be reformulated as

$$\Delta p_{\text{det},\mathcal{H}} = \frac{1}{\sqrt{2} \dot{s}_{\mathcal{H}}(\hat{p})} \left( \frac{1}{\sqrt{\text{SNR}_1 \cdot L}} + \frac{1}{\sqrt{\text{SNR}_2 \cdot L}} \right)$$

$$= \frac{1}{\sigma_{\mathcal{H}}} \Delta p_{\text{det}} \quad (3.19)$$

and

$$\Delta p_{\text{res},\mathcal{H}} = \frac{1}{\sigma_{\mathcal{H}} \dot{s}(\hat{p})} \sqrt{\frac{2}{\kappa_{1,\mathcal{H}}(\hat{p})} - \frac{1}{\|\hat{2}(p)\|^2}} \left( \frac{1}{\sqrt{\text{SNR}_1 \cdot L}} + \frac{1}{\sqrt{\text{SNR}_2 \cdot L}} \right)$$

$$= \frac{1}{\sigma_{\mathcal{H}}} \Delta p_{\text{res}} \quad (3.20)$$

It is interesting to note that the detection and resolution thresholds for the
3.2 Array Bounds Based on the Extended Array Manifolds

extended and the spatial array manifold curves are connected in such a way. The connection is demonstrated better in Equations (3.21) and (3.22). Assuming that $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$, by rearranging Equations (3.19) and (3.20), the minimum number of snapshots required to detect $(L_{\text{det}, \mathcal{H}})$ and resolve $(L_{\text{res}, \mathcal{H}})$ the two sources can be expressed as follows.

$$L_{\text{det}, \mathcal{H}} = \frac{2}{s_{\hat{H}}^2 (\hat{p})} \frac{1}{\text{SNR}} \frac{1}{\Delta p^2} = \frac{1}{\sigma_k^2} L_{\text{det}}$$  \hspace{1cm} (3.21)$$

and

$$L_{\text{res}, \mathcal{H}} = \frac{32}{s_{\hat{H}}^4 (\hat{p}) \hat{\kappa}_{1.\mathcal{H}}^2 (\hat{p})} \frac{1}{\text{SNR}} \frac{1}{\|h(\hat{p})\|^2} = \frac{1}{\sigma_k^2} L_{\text{res}}$$ \hspace{1cm} (3.22)$$

where $L_{\text{det}}$ and $L_{\text{res}}$ are the corresponding bounds when the spatial array manifolds are employed.

This relationship between the thresholds for the spatial and the extended array manifold curves (given in Equations (3.21) and (3.22)) is due to the fact that the radius of the hypersphere on which the extended manifold lies is greater than the corresponding radius of the “spatial” hypersphere. Thus, if the angular separation between two spatial manifold vectors $a_1$ and $a_2$ is equal to the angular separation between the associated extended manifold vectors $h_1$ and $h_2$, then

$$\|h_2 - h_1\| > \|a_2 - a_1\|$$

Hence, the extended array manifolds can accommodate a larger radius of the uncertainty sphere, i.e. more detrimental noise effects, and still be able to detect and resolve the two different sources. Note that in this analysis we assume that the signal to noise ratio is the same in the case of a system described by the spatial manifold and a system described by an extended manifold. This is justified since the extension is due to a change in the system architecture and not the noise in the channel or the thermal noise of the electronics/antennas in the re-
In addition, we should mention that we consider here the same functional dependence of the radius of the uncertainty sphere on the noise power as in [34].

From a communications system perspective, this is better demonstrated if one considers the STAR manifold as an example of a conformal extended array manifold. In the case of the STAR manifold, as can be seen from Equation (2.40), \( \sigma_A^2 = N_c \), where \( N_c \) is the length of the PN Sequence, also known as the Processing Gain of the spread spectrum system, which is actually a gain in the SNR of the system. Therefore, the effect of the PN sequence is an increase in the SNR which can allow for a reduction of the total number of snapshots \( L \), so that the product \( \text{SNR} \times L \) remains constant. Note, however, that the number of snapshots \( L_{\text{det}} \) or \( L_{\text{res}} \) refers to the number of available observation vectors \( \mathbf{x}[n], \forall n \) at point C of Figure 2.2. In the case of an asynchronous CDMA system fitting the model presented in Figure 2.2 (properly modelled in [33] using the STAR manifold), each of the vectors \( \mathbf{x}[n] \) contains \( 2NN_c \) samples, while the vectors \( \mathbf{x}(t_k), \forall k \) at point B of the system described in Figure 2.1 contain only \( N \). However, at each sampling instance, only half of these values contained in each \( \mathbf{x}[n] \) were actually sampled at Point A of Figure 2.2, the rest were just duplicated by remaining for two symbol periods in the Tapped Delay Line, which in this case is the Rx-LPU. This implies that the asynchronous CDMA system requires exactly the same number of sample values (at point A) of the analog baseband signal as the reference system of Figure 2.1, namely

\[
\frac{L_{\text{det}} \times (N \times N_c)}{\text{System of Figure 2.2}} = \frac{L_{\text{det}} \times N}{\text{System of Figure 2.1}} \tag{3.23}
\]

This may be explained by the fact that the only thing that actually matters for the array system is the energy carried by each sample at Point A of the two Figures.

The bounds illustrated in Figures 3.12 - 3.19 have been produced based on the same scenarios as those of Figures 3.4 - 3.11, but using the STAR manifold and a PN-code of length \( N_c = 15 \). By comparing the two sets of figures, one can
observe that the only difference between them the number of snapshots required for the detection and the resolution of the two sources. Finally, in Figure 3.20 the maximum required number of snapshots for detecting the two sources is plotted as a function of the length $N_c$ of the PN-code. As was expected from Equations (3.21) - (3.22), the shape of the curve of Figure 3.20 is a hyperbola.
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.12: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ for the array of Figure 3.2

Figure 3.13: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ for the array of Figure 3.2
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.14: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ of the array of Figure 3.2

Figure 3.15: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ for the array of Figure 3.2
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.16: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ of the array of Figure 3.7

Figure 3.17: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ for the array of Figure 3.7
Figure 3.18: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ of the array of Figure 3.7

Figure 3.19: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, $N_c = 15$ for the array of Figure 3.7
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.20: Maximum required snapshots for the detection of two sources, \( \Delta \theta = \Delta \phi = 0.5^\circ \), \( \text{SNR} = 20\text{dB} \), for the array of Figure 3.7 as a function of the length of the PN-code

### 3.2.2 Array Bounds of “Non-Conformal” Extended Array Manifolds

In the case of extended array manifolds emerging from non-conformal complex mappings of the associated spatial array manifolds, the same line of reasoning as the one presented for the spatial array manifolds in Section 3.2.1 will lead to Equations (3.24) (3.25) given below.

\[
\Delta p_{\text{det}, \mathcal{H}} = \frac{1}{\sqrt{2} \hat{s}_{\mathcal{H}} (\hat{p})} \left( \frac{1}{\sqrt{\text{SNR}_1} \cdot L} + \frac{1}{\sqrt{\text{SNR}_2} \cdot L} \right)
\]

(3.24)

and

\[
\Delta p_{\text{res}, \mathcal{H}} = \frac{1}{\hat{s}_{\mathcal{H}} (\hat{p})} \sqrt{\frac{2}{\kappa_{1, \mathcal{H}} (\hat{p}) - \frac{1}{\|h(\rho)\|^2}}} \left( \frac{1}{\sqrt{\text{SNR}_1} \cdot L} + \frac{1}{\sqrt{\text{SNR}_2} \cdot L} \right)
\]

(3.25)

The only difference between these two expressions and Equations (3.19) (3.20) is that the group of Equations (3.18) is not valid and, thus, no simple relationship
between the bounds for the spatial and the extended manifolds can be derived.

The necessary geometric parameters can be calculated using the procedure and the expressions presented in Chapter 2. Once the required geometric properties \( s_A (\hat{p}) \) and \( \kappa_1 (\hat{p}) \) of the associated spatial array manifold curve have been calculated and the matrix \( A (\hat{p}) \) of the complex mapping producing the extended manifold vector has been formed, the corresponding properties \( s_H (\hat{p}) \) and \( \kappa_1, H (\hat{p}) \) can be estimated directly using Equations (2.27) and (2.35)-(2.37). Note that

\[
\hat{\kappa}_{1,H} (\hat{p}) = \kappa_{1,H} (\hat{p}) \sin \zeta_H (\hat{p}) \tag{3.26}
\]

and

\[
\sin \zeta_H \triangleq \sqrt{1 - \left| u_{2,H} (\hat{p}) u_{1,H} (\hat{p}) \right|^2} \tag{3.27}
\]

Next, two representative examples of detection and resolution bounds for “non-conformal” extended array manifold vectors will be given. The first is based on the manifold vector of a non-omnidirectional array consisting of perfect dipoles. The second is based on the POLAR manifold vector.

**Performance bounds for an array of perfect dipole antennas**

The array manifold vector for an array of perfect dipole antennas is given by

\[
h_d (\theta, \phi) = g_d (\phi) \mathbf{a} (\theta, \phi) \tag{3.28}
\]

where \( \theta \) and \( \phi \) are the directional parameters, \( \mathbf{a} (\theta, \phi) \) is the spatial array manifold vector and \( g_d (\phi) \) is the normalised gain pattern of a perfect dipole, given by

\[
g_d (\phi) = \cos^2 \phi \tag{3.29}
\]

The gain pattern of a single antenna of the array is depicted in Figures 3.21 and 3.22. For the entire arrays described in Figures 3.2 and 3.7 the gain patterns are shown in 3.23 and 3.24 respectively. Note that the array pattern is defined as

\[
g (\theta, \phi) = \left| \mathbf{1}_N^T \mathbf{a} (\theta, \phi) \right| \tag{3.30}
\]
Figure 3.21: Normalised gain pattern for an ideal dipole

Figure 3.22: Cross sections of the gain pattern of an ideal dipole
Figure 3.23: Gain pattern for the array of Figure 3.2 when each element is an ideal dipole

Figure 3.24: Gain pattern for the array of Figure 3.7 when each element is an ideal dipole
The required geometric parameters for the estimation of the detection and resolution thresholds in terms of the cone-angle parameters $\alpha$, $\beta_0$ and $\Theta_0$ are given below. Note that $\beta_0$ and $\Theta_0$ are fixed parameters and for that reason the dependence of the geometric parameters on them will be dropped for the sake of simplicity. Moreover, $\dot{x}$ will denote differentiation with respect to the variable $\alpha$.

\[ g_d(\alpha) = \sigma_h(\alpha) = 1 - \cos^2 \alpha - \cos^2 \beta_0 \quad (3.31a) \]

\[ \sigma_h(\alpha) = 2 \cos \alpha \sin \alpha \quad (3.31b) \]

\[ d_0(\alpha) = \frac{d s_H(\alpha)}{d \alpha} = \| \frac{d h_d(\alpha)}{d \alpha} \| = \sqrt{\sigma^2_h(\alpha) N + \sigma^2_h(\alpha) \dot{s}_A^2(\alpha)} \quad (3.31c) \]

\[ d_1 = \frac{d^2 s_H(\alpha)}{d \alpha^2} = \frac{N \sigma_h(\alpha) \dot{\sigma}_h(\alpha) + \sigma_h(\alpha) \sigma_h(\alpha) \dot{s}_A^2(\alpha) + \sigma_h^2(\alpha) \dot{s}_A(\alpha) \dot{\dot{s}_A}(\alpha)}{d_0} \quad (3.31d) \]

\[ e_0(\alpha) = \dot{h}_d(\alpha) \quad (3.31e) \]

\[ e_1(\alpha) = \ddot{h}_d(\alpha) \quad (3.31f) \]

where

\[ s_A^2 = \left\| \frac{d a(\alpha)}{d \alpha} \right\|^2 \]

is the square of the rate of change of the arc length $s_A(\alpha)$ of the spatial $\alpha$-curve (see Equation (C.2) in Appendix C).

The resulting detection and resolution bounds for the two array geometries studied earlier (see Figures 3.2 and 3.7) can be seen in Figures 3.25 - 3.32.

Note that in Figures 3.27 and 3.28 there is a variation of the threshold with regards to the azimuth, but is not easily distinguished in the figures due to the log scale.
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.25: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2 when each antenna is a perfect dipole.

Figure 3.26: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2 when each antenna is a perfect dipole.
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.27: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, $\text{SNR} = 20\text{dB}$ of the array of Figure 3.2 when each antenna is a perfect dipole.

Figure 3.28: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, $\text{SNR} = 20\text{dB}$ for the array of Figure 3.2 when each antenna is a perfect dipole.
Figure 3.29: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.7 when each antenna is a perfect dipole.

Figure 3.30: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.7 when each antenna is a perfect dipole.
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.31: Contour plot of the resolution capabilities for two sources, $\Delta\theta = \Delta\phi = 0.5^\circ$, SNR = 20dB of the array of Figure 3.7 when each antenna is a perfect dipole.

Figure 3.32: Required snapshots for the resolution of two sources, $\Delta\theta = \Delta\phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.7 when each antenna is a perfect dipole.
Performance bounds based on the POLAR array manifold

The analysis in the case of the POLAR manifold is similar to that of the spatial array manifold of an array of perfect dipoles. The polarisation parameters assume the following values.

<table>
<thead>
<tr>
<th>$V_x$</th>
<th>$V_y$</th>
<th>$V_z$</th>
<th>$\gamma$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
<td>0.1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The gain pattern for the arrays of Figures 3.2 and 3.7 can be found in Figures 3.33 and 3.34 respectively.
The required parameters for the estimation of the detection and resolution thresholds are given next.

\[ A_P = \mathbb{I}_N \otimes \hat{z}_P \]  \hspace{1cm} (3.32a)

\[ \hat{z}_P = \text{diag} \{ [V_x, V_y, V_z] \} \left[ \frac{1}{\cos \phi} \hat{u}_g, \hat{u}_p, \cos \gamma, \sin \gamma \exp \{ j\eta \} \right]^T \]  \hspace{1cm} (3.32b)

\[ \hat{h}_P = A_P \tilde{a} \]  \hspace{1cm} (3.32c)

and the remaining parameters are estimated using Equations (3.31). The detection and resolution bounds for both arrays are given in Figures 3.35 - 3.42.
Figure 3.35: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2 when each element is polarisation sensitive.

Figure 3.36: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB for the array of Figure 3.2 when each element is polarisation sensitive.
Figure 3.37: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, of the array of Figure 3.2 when each element is polarisation sensitive.

Figure 3.38: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, for the array of Figure 3.2 when each element is polarisation sensitive.
3.2 Array Bounds Based on the Extended Array Manifolds

Figure 3.39: Contour plot of the detection capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, for the array of Figure 3.7 when each element is polarisation sensitive.

Figure 3.40: Required snapshots for the detection of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, for the array of Figure 3.7 when each element is polarisation sensitive.
Figure 3.41: Contour plot of the resolution capabilities for two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, of the array of Figure 3.7 when each element is polarisation sensitive

Figure 3.42: Required snapshots for the resolution of two sources, $\Delta \theta = \Delta \phi = 0.5^\circ$, SNR = 20dB, for the array of Figure 3.7 when each element is polarisation sensitive
3.3 Performance Bounds in Terms of the Manifold Surface

The study of the theoretical performance bounds in the cases of both “conformal” and “non-conformal” extended array manifolds, presented in the previous sections, has been based on the circular approximation of an array manifold curve passing from the two points of interest on the array manifold surface. However, as has been shown previously, it is not always straightforward to obtain a manifold curve passing through two specific points of interest and in most cases a coordinate transformation is required, namely the transformation from the \((\theta, \phi)\) parametrization to the \((\alpha, \beta)\) one. This results in complicated expressions which require a heavy computational burden both for calculating the analytical expressions of the various geometric parameters and for the numerical evaluation of these expressions.

For the aforementioned reasons, in this section, a new, alternative, approach in the estimation of the detection bound for array systems is presented. This approach has to satisfy the following constraints

- It has to be able to handle arbitrary array manifolds, not necessarily curves and not necessarily lying on a complex sphere.
- The resulting performance bounds have to agree with the ones calculated using the manifold curves based method presented so far.
- It needs to be able to deal with all possible scenarios (e.g. sources with elevation angle \(\phi = 0\) or sources with the same azimuth \(\theta\)).

and is presented next.

3.3.1 Detection threshold

Let us assume that an array of arbitrary geometry operates in the presence of two sources parameterized using the generic parameters \((p, q)\). Note that \(p\) and \(q\) may
be directional parameters or any other parameter of interest. These two sources correspond to two distinct extended array manifold vectors $h_1 \triangleq h(p_1, q_1)$ and $h_2 \triangleq h(p_2, q_2)$. The aim is to calculate the theoretical thresholds for the detection of these two sources by the antenna array in a manner independent of any array manifold curve passing through the two points $h_1(p_1, q_1)$ and $h_2(p_2, q_2)$ on the extended array manifold surface.

The correct number of sources can be detected if and only if the dimension of the subspace spanned by the manifold vectors equals the number of sources or, equivalently, if the set of manifold vectors is not linearly dependent. In the case of two sources, linear dependency corresponds to the two manifold vectors $h_1$ and $h_2$ being co-linear. However, assuming that $h_1$ and $h_2$ are not co-linear, i.e.

$$\exists c_1 \in \mathbb{C} : h_1 = c_1 h_2$$

(3.33)

the addition of random effective-noise vectors $n_{e,i}$ and $n_{e,2}$ of variance $\sigma_{e,i}^2$, $i = 1, 2$ to them may result in co-linear vectors, i.e.

$$\exists c_2 \in \mathbb{C} : h_1 + n_{e,1} = c_2 (h_2 + n_{e,2})$$

(3.34)

where $\sigma_{e,i} \triangleq \sqrt{\mathbb{E} \{ \| n_{e,i} \|^2 \}}$ is the radius of the uncertainty sphere of the effective-noise (see [31]). Note that $\mathbb{E} \{ \cdot \}$ stands for the expected value of a variable.

Of course, there are many ways in which the uncertainty spheres can perturb the manifold vectors in order to render them co-linear. However, the most restrictive case, the one which requires the smallest cumulative length of the uncertainty spheres and consequently the smallest noise power occurs when only one of the manifold vectors is “projected” on the 1-dimensional subspace spanned by the other. Hence, with reference to Figure 3.43 the detection threshold occurs when the distance $\sigma_e = \sigma_{e,1} + \sigma_{e,2}$ is equal to the distance $d_{2,1}$ of the second (say) manifold vector $h_2$ from its projection $\mathbb{P}_{h_1} h_2$ on $h_1$, where

$$\mathbb{P}_{h_1} = h_1 (h_1^H h_1)^{-1} h_1^H$$

(3.35)

$^2$See [31] for a definition of the effective noise.
However, if the two sources are close together, then the following approximation is valid

\begin{equation}
  d_{2,1} = \| h_2 - P_{h_1} h_2 \| \preceq \| h_2 - h_1 \| \tag{3.36}
\end{equation}

Therefore the two sources can be detected if and only if

\begin{equation}
  \| h_2 - h_1 \| \geq \sigma_e \tag{3.37}
\end{equation}

By taking a first order Taylor expansion of $h_2$ around $h_1$, Equation (3.37) is reformulated as

\begin{equation}
\begin{aligned}
  \left\| h_1 + \Delta p \frac{\partial h(p,q)}{\partial p} \bigg|_{p=p_1, q=q_1} + \Delta q \frac{\partial h(p,q)}{\partial q} \bigg|_{p=p_1, q=q_1} - h_1 \right\| & \geq \sigma_e \Rightarrow \\
  \left\| \Delta p \frac{\partial h(p,q)}{\partial p} \bigg|_{p=p_1, q=q_1} + \Delta q \frac{\partial h(p,q)}{\partial q} \bigg|_{p=p_1, q=q_1} \right\| & \geq \sigma_e \Rightarrow \\
  \left[ \begin{array}{c}
  \dot{h}_p(p_1, q_1) \\
  \dot{h}_q(p_1, q_1)
\end{array} \right]_{T_H} \left[ \begin{array}{c}
  \Delta p \\
  \Delta q
\end{array} \right]_{\hat{\Delta}H} & \geq \sigma_e \Rightarrow \\
  \Delta p^T T_H^T \Delta p & \geq \sigma_e^2 \Rightarrow \\
  \Delta p^T \text{Re} \{ T_H^T T_H \} \Delta p & \geq \sigma_e^2 \Rightarrow \\
  \Delta p^T G_H \Delta p & \geq \sigma_e^2
\end{aligned}
\end{equation}

Figure 3.43: Graphical representation of the minimum distance between the two manifold vectors
Therefore, the detection threshold occurs when the first fundamental form $I$ of the extended array manifold surface is equal to the square of the sum of the uncertainty spheres, that is

$$I = \Delta p^T G_H \Delta p = \sigma_e^2$$

Equation (3.39)

It is important to note that in the case of two sources lying on the same $p$-curve, the $q$ parameter is the same for both sources and for that reason $\hat{h}_q = 0$. Hence, Equation (3.39) reduces to

$$\Delta p_i^2 \hat{h}_1 \hat{h}_1 = \sigma_e^2$$

Equation (3.40)

which is equal to Equation (3.2) since $\hat{h}_1 \hat{h}_1 = \hat{s}_H (p)$.

This approach produces identical results with the methods presented in [31] when applied to sources lying on a known manifold curve. In addition, it can be applied to any extended array manifold and does not require a manifold curve passing from the points of interest (i.e. the points corresponding to the two sources) on the array manifold surface. Therefore, it is much more computationally efficient than the coordinates transformation method which was implemented in Section 3.1 to calculate the detection bounds for 3-dimensional array geometries because no coordinates translation is necessary.

In Figures 3.44 - 3.49 the difference in the predicted number of required snapshots for two sources with 0.5° separation in the azimuth and elevation is plotted for the arrays of Figures 3.2 and 3.7, for omnidirectional array elements, perfect dipole array elements and array elements sensitive to the polarisation of the incoming signal. In all cases $\text{SNR} = 20\text{dB}$. It can be seen that the two methods provide almost identical results. The small discrepancies arise due to the different approximations upon which the two methods rely. The curves method approximates a manifold curve with a circular arc, while the surfaces method approximates the manifold surface with a flat surface. These two approximation are of different accuracy, so these small deviations in the results appear.
Figure 3.44: Difference in the estimated number of snapshots for detection between the two methods, for the array of Figure 3.2 and omnidirectional elements are assumed.

Figure 3.45: Difference in the estimated number of snapshots for detection between the two methods, for the array of Figure 3.7 and omnidirectional elements are assumed.
Figure 3.46: Difference in the estimated number of snapshots for detection between the two methods, for the array of Figure 3.2 and the antennas are assumed to be perfect dipoles.

Figure 3.47: Difference in the estimated number of snapshots for detection between the two methods, for the array of Figure 3.7 and the antennas are assumed to be perfect dipoles.
3.3 Performance Bounds in Terms of the Manifold Surface

Figure 3.48: Difference in the estimated number of snapshots for detection between the two methods, for the array of Figure 3.2 and the antennas are assumed to be polarisation-sensitive.

Figure 3.49: Difference in the estimated number of snapshots for detection between the two methods, for the array of Figure 3.7 and the antennas are assumed to be polarisation-sensitive.
3.4 Summary

In this chapter the problem of theoretical array performance bounds was addressed. Initially, the existing theoretical bounds for linear and planar arrays were extended for the case of random 3-dimensional array geometries. Then, the concept of the extended array manifolds was employed to carry these results to more complex array systems. Finally, a new approach for the calculation of the detection threshold was presented which depends not on the geometry of manifold curves, but on the geometry of manifold surfaces. This approach was shown to be a generalisation of the existing curves method and, in addition, removes the singularities of the curves method at the boundary of the manifold.
Chapter 4

Hyperhelical Manifold Curves

The differential geometry approach, which studies the array manifold as a geometric object embedded in a multidimensional complex space has led to a number of significant results regarding the performance analysis of array systems. The problem of ambiguities in array processing has been analysed and linked with array manifold geometric properties in [36, 37]. In [34], theoretical lower bounds regarding the detection, resolution and accuracy capabilities of linear array systems have derived as function of the intrinsic geometry of the spatial array manifold. Moreover, in Chapter 3 these results were extended to the case of random 3-dimensional array geometries and then to array systems modelled using the extended array manifolds enabling for a comprehensive study of practical array systems. Finally, in [7] geometric concepts have been utilised to tackle the array design problem in an optimal way.

Although the differential geometric approach has been shown to be fruitful, the geometric analysis of the array manifolds for a purely random 3-dimensional array geometry can be extremely complex due to the high degree of non-linearity of the corresponding geometric objects. This, in turn, renders the derivation of the aforementioned theoretical results an extremely arduous matter and usually one is compelled to resort to approximations. However, in [31] a specific class of spatial array manifold curves have been identified (array manifolds depending on a single parameter of interest), the analysis of which is greatly simplified. In detail, array manifold curves belonging in this class have constant curvatures
throughout the length of the curve, which implies that they are of hyperhelical shape. In the case of hyperhelical manifold curves, analytic expressions for the most significant theoretical results and bounds have been derived [31].

In addition, in Chapter 2 the concept of hyperhelix was carried over to the extended array manifolds, where it was shown (see Theorem 1) that if the spatial array manifold curve has constant curvatures and the complex mapping which produces the extended array manifold curve is conformal, then the extended array manifold curve has constant curvatures as well. Therefore, if the conditions of this theorem are met, the analysis of the extended array manifold curves is simplified as well.

These properties render the hyperhelical manifold curves extremely useful. Given their importance, it is interesting to determine which array geometries can, in theory, give rise to such curves. Due to Theorem 1, it is sufficient to determine which array geometries give rise to hyperhelical spatial array manifold curves, since if the spatial array manifold curve does not have constant curvatures, it is very unlikely (but not impossible) that the extended array manifold curve will be a hyperhelix. Note that an extended manifold curve, derived as a complex mapping of a non-hyperhelical manifold curve, may still be a hyperhelix, provided that the complex mapping is not conformal and that its effect is such that turns non-hyperhelical manifold curve into a hyperhelical one. Although very rare to appear in practice, such mappings may exist and determining their nature is an interesting challenge which is closely related to the array design problem.

Hence, the purpose of this chapter is to investigate the necessary and sufficient conditions for the existence of hyperhelical spatial array manifold curves in an array of a given geometry. (This, of course, is related to the existence of extended hyperhelical manifold curves as well.) After these conditions have been identified, the next goal is to determine all the possible array geometries or classes of array geometries, which can give rise to hyperhelices.
4.1 Hyperhelical Space Curves Embedded in $\mathbb{C}^N$

Prior to presenting the main theoretical results of this chapter, a short definition of some notions of differential geometry, which will be used frequently in the rest of this chapter, will be given.

A *regular parametric representation* of a curve $\mathcal{A}$ embedded in a $N$-dimensional complex space ($\mathbb{C}^N$) is a complex valued vector function

$$\mathbf{a}(p) \in \mathbb{C}^{N \times 1}, \; p \in I \subseteq \mathbb{R}$$

(4.1)

of the parameter $p$, defined in an interval $I \subseteq \mathbb{R}$, with the following properties:

1. $\mathbf{a}(p)$ is of class $C^1$ in $I$

$$\dot{\mathbf{a}}(p) \triangleq \frac{d\mathbf{a}(p)}{dp} \neq 0, \; \forall p \in I$$

(4.2)

A real valued function $p = p(p) : I \rightarrow I$ is an *allowable change of parameter* for the expression of $\mathcal{A}$ if

$$p(p) \text{ is of class } C^1 \text{ in } I$$

$$\frac{dp}{dp} \neq 0, \; \forall p \in I$$

(4.3)

Two regular representations $\mathbf{a}(p)$ and $\mathbf{a}(\tilde{p})$ are *equivalent* if and only if there exists an allowable change of parameter $p = p(p)$ such that

$$\mathbf{a}(p(p)) = \mathbf{a}(p), \; \forall p \in I$$

What has to be pointed out is that a curve $\mathcal{A}$ may have many equivalent regular parametric representations, but the properties of the curve are independent of the parameter.

Assume that $\mathbf{a}(p)$ is a regular parametric representation of the curve $\mathcal{A}$ and

\footnote{A function is said to be of class $C^1$ if its first derivative exists and is continuous}
let \( s = s(p) \) define the arc length of the curve, as it is measured from a given point \( \mathbf{a}(p_0) \) on the curve, that is

\[
    s = s(p) = \int_{p_0}^{p} \| \dot{\mathbf{a}}(\xi) \| \, d\xi
\]  (4.4)

However from Equations (4.2) and (4.3), \( \dot{s}(p) \equiv \| \dot{\mathbf{a}}(p) \| \) is continuous and non-zero \( \forall p \in I \). Thus, \( s = s(p) \) is an allowable change of parameter on \( I \) and \( \mathbf{a} = \mathbf{a}(s) \) is a regular parametric representation of \( \mathcal{A} \). Furthermore,

\[
    \| \mathbf{a}'(s) \| \triangleq \left\| \frac{d\mathbf{a}(s)}{ds} \right\| = 1
\]

so that \( \mathbf{a} = \mathbf{a}(s) \) is called a \textit{natural representation} of \( \mathcal{A} \).

A very important concept of Differential Geometry related to curves is that of the curvatures of a curve. According to the Fundamental Uniqueness Theorem by Gauss, a space curve expressed in a \textit{natural representation}, (i.e. in term of its arc length) is uniquely defined by its curvatures, except for its position in space. For a curve \( \mathcal{A} \) of the form of Equation (4.1) defined in \( I = [p_{\text{min}}, p_{\text{max}}] \) and an allowable change of parameter of the form of Equation (4.4), an equivalent, natural representation is the following

\[
    \mathbf{a} = \mathbf{a}(s) \in \mathbb{C}^{N \times 1}, \quad s(p_{\text{min}}) = 0 \leq s \leq l_m \triangleq s(p_{\text{max}})
\]  (4.5)

A curve embedded in an \( N \)-dimensional complex space has \( 2N - 1 \) non-zero curvatures \( \kappa_1, \kappa_2, \ldots, \kappa_{2N-1} \). If, however, this curve is limited to a subspace \( \mathbb{C}^u \) of \( \mathbb{C}^N \), then only \( d - 1 = 2u - 1 \) non-zero curvatures can be defined. A (real) example of this is a curve lying on the plane \( x = y \) of \( \mathbb{R}^3 \). Although the dimensionality of the real space is 3, the curve is limited on a plane, the dimensionality of which is obviously equal to 2 and, hence, only \( \kappa_1 \) is non-zero.

The curvatures of a space curve \( \mathcal{A} \) embedded in a subspace of \( \mathbb{C}^N \) are defined using the concept of the Frenet Frame, which is a set of \( d \) complex unit vectors
\textbf{4.1 Hyperhelical Space Curves Embedded in } \mathbb{C}^N \textbf{118}

\(\mathbf{u}_i(s), \ i = 1, \ldots, d,\text{ which are attached at every point } s \text{ of the curve and serve as a local coordinate system.}

The Frenet vectors and the curvatures are given by

\[
\begin{align*}
\mathbf{u}_1(s) &= \mathbf{u}'(s) \quad \kappa_1 = \|\mathbf{u}'(s)\| \\
\mathbf{u}_2(s) &= \frac{\mathbf{u}'_1(s)}{\kappa_1(s)} \quad \kappa_2(s) = \|\mathbf{u}'_2(s) + \kappa_1(s)\mathbf{u}_1(s)\| \\
\vdots \quad \ddots \\
\mathbf{u}_i(s) &= \frac{\mathbf{u}'_{i-1}(s) + \kappa_{i-2}(s)\mathbf{u}_{i-2}(s)}{\kappa_{i-1}(s)} \quad \kappa_i(s) = \|\mathbf{u}'_i(s) + \kappa_{i-1}(s)\mathbf{u}_{i-1}(s)\| \\
\vdots \quad \ddots
\end{align*}
\]

(4.6)

The curvatures are in the general case functions of the arc length \(s\). We define as \textit{hyperhelix} the space curve \(\mathcal{A}\) which has constant curvatures, that is

\[
\kappa_i(s) = \kappa_i, \ i = 1, \ldots, d
\]

(4.7)
4.2 Natural Representation of Hyperhelices

So far, in [31] it has been shown that the spatial array manifold curves of linear arrays and the elevation and cone angle spatial manifold curves of planar arrays are hyperhelices. The regular parametric representations of these two classes of curves present some distinct similarities. Based on this observation, the question naturally arises of whether there is a more general equation for the representation of a hyperhelical manifold curve embedded in $\mathbb{C}^N$ and whether this equation is unique, i.e.

- Is there an equation which may serve as a general regular parametric representation of a hyperhelical manifold curve (not necessarily an array manifold curve but any space curve) and which encompasses the known formula of hyperhelical array manifold curve as a special case?

- If such a representation does exist, is it unique?

Theorem 3, which follows, is the first step towards answering the previous questions. It provides a general formula for a natural representation of a hyperhelical manifold curve in $\mathbb{C}^N$ and asserts that this representation in terms of the arc length $s$ of the curve is unique.

**Theorem 3.** Let a space curve $A$ be embedded in a $u$-dimensional complex space $U \equiv \mathbb{C}^u$ and assume that $U$ is the space of minimum dimensionality which contains $A$. Then, $A$ is a hyperhelix, that is the curvatures of $A$ are constant, if and only if there is a constant real vector $c \in \mathbb{R}^{d \times 1}$ and a constant complex vector $v \in \mathbb{C}^{d \times 1}$, with

\[
\begin{align*}
    d &= 2u \\
    \|v \odot c\| &= 1 \\
    1^T_N c &= 0
\end{align*}
\]

such that

\[
a(s) = v \odot \exp \{-jc \cdot s\}
\]
is a natural representation of $\mathcal{A}$.

**Proof.** Throughout this proof the notation $x'$ will denote differentiation of the quantity $x$ with respect to the arc length $s$.

Note that Equation (4.8b) guarantees that Equation (4.9) is a natural representation, since

$$
\|a'(s)\| = \|v \odot \xi\| = 1
$$

**Forward:** Let us assume that Equation (4.9) is a natural representation of the curve $\mathcal{A}$. Then, it is sufficient to show that its curvatures are independent of the arc length $s$. Based on Equation (4.6), the lower order coordinate vectors and curvatures are given below.

$$
\begin{align*}
\mathbf{u}_1(s) &= a(s)' = -j \xi \odot a(s) \\
\mathbf{u}_1'(s) &= -\xi^2 \odot a(s) \\
\mathbf{u}_2(s) &= -\frac{1}{\kappa_1} \xi^2 \odot a(s) \\
\mathbf{u}_2'(s) &= j \xi^3 \odot a(s) \\
& \vdots
\end{align*}
$$

(4.10)

$$
\kappa_i = \left\| \mathbf{u}_i' + \kappa_{i-1} \cdot \mathbf{u}_{i-1} \right\| = \frac{1}{\kappa_1 \kappa_2 \cdots \kappa_{i-1}} \cdot \left\| v \odot \sum_{n=1}^{\left\lfloor \frac{i}{2} \right\rfloor + 1} (-1)^{n-1} b_{i-1,n} \xi^{i-2n+3} \right\| 
$$

(4.12)

where

$$
b_{i,n} = b_{i-1,n} + \kappa_{i-1}^2 \cdot b_{i-2,n-1}, \quad i \geq 1
$$

(4.13)
4.2 Natural Representation of Hyperhelices

with

\[ b_{i,1} = 1, \ i \geq 1 \]
\[ b_{2,2} = \kappa_1^2 \]  

(4.14)

The proof of Equations (4.11) - (4.14) can be found in Appendix D. Since the curve is embedded in a \( u \)-dimensional complex space, these formulae are valid for \( i = 1, \ldots, d - 1 = 2u - 1 \) since the \( d \)-th curvature is zero for any curve, not only a hyperhelix, constrained in a \( u \)-dimensional complex space. The important characteristic of Equation (4.12) is that it is independent of the parameter \( s \). Hence, by the definition of hyperhelix, curve \( \mathcal{A} \) is of hyperhelical shape.

**Converse:** Let us assume now that \( \mathcal{A} \) is a hyperhelix, embedded in the \( u \)-dimensional complex space \( \mathcal{U} \). Since by assumption \( \mathcal{U} \) is the space of minimum dimensionality that contains \( \mathcal{A} \), the latter has \( d = 2u \) constant curvatures \( \kappa_1, \kappa_2, \ldots, \kappa_d \), of which the first \( d - 1 \) are non-zero and of course \( \kappa_d = 0 \). Let us consider the set \( \mathcal{S}_d^\infty \) of all the possible hyperhelices embedded in the same complex space \( \mathcal{U} \). Each hyperhelix \( \mathcal{A}^{\text{helix}}_\Sigma \) in \( \mathcal{S}_d^\infty \) is uniquely defined, according to the Fundamental Uniqueness theorem by Gauss, by a vector \( \kappa = [\kappa_1, \kappa_2, \ldots, \kappa_{d-1}, 0]^T \) of curvatures.

Consider now the set \( \mathcal{S}_d^c \) of those hyperhelices which can be expressed by a natural representation in the form of Equation (4.9). It was shown previously that \( \mathcal{S}_d^c \subseteq \mathcal{S}_d^\infty \). The proof will be complete if it is shown that \( \mathcal{S}_d^c = \mathcal{S}_d^\infty \).

To that end, let us choose one of the members \( \mathcal{A}^{\text{helix}}_\Sigma \) of \( \mathcal{S}_d^\infty \), which is defined by a vector \( \kappa = [\kappa_1, \kappa_2, \ldots, \kappa_{d-1}, 0]^T \) of constant curvatures. The objective is to find a constant real vector \( \underline{c} \in \mathbb{R}^{N \times 1} \) and a complex constant vector \( \underline{v} \in \mathbb{C}^{N \times 1} \), such that the curve \( \mathcal{A} \) having as a natural representation the vector function \( a(s) = \underline{v} \odot \exp \{ -j\underline{c} \cdot s \} \) has the same curvatures as \( \mathcal{A}^{\text{helix}}_\Sigma \).

Let us write down the Frenet-Serret formulae [21] which connect the coordi-
nate vectors \( \mathbf{u}_i, \ i = 1, \ldots, d \) and curvatures \( \kappa_i, \ i = 1, \ldots, d \) of a space curve.

\[
\left( \mathbf{U}(s)^T \right)' = \mathbf{C}(s) \mathbf{U}(s)^T \tag{4.15}
\]

where

\[
\mathbf{C}(s) = \begin{bmatrix}
0 & \kappa_1(s) & 0 & \ldots & 0 & 0 \\
-\kappa_1(s) & 0 & \kappa_2(s) & \ldots & 0 & 0 \\
0 & -\kappa_2(s) & 0 & \ldots & 0 & 0 \\
: & : & : & \ddots & : & : \\
0 & 0 & 0 & \ldots & 0 & \kappa_{d-1}(s) \\
0 & 0 & 0 & \ldots & -\kappa_{d-1}(s) & 0
\end{bmatrix} \tag{4.16}
\]

is the Cartan matrix and

\[
\mathbf{U}(s) = [\mathbf{u}_1(s), \ldots, \mathbf{u}_d(s)] \tag{4.17}
\]

Let us denote as

\[
\mathbf{Y} \triangleq \mathbf{U}^T = \begin{bmatrix}
y_1(s) \\
y_2(s) \\
\vdots \\
y_N(s)
\end{bmatrix} \tag{4.18}
\]

where

\[
y_i(s) \triangleq [U_{i1}(s), U_{i2}(s), \ldots, U_{id}(s)]^T
\]

the \( i \)-th column of \( \mathbf{U}^T \). Then, Equation (4.15) can be re-written as

\[
\mathbf{Y}(s)' = \mathbf{C}(s) \mathbf{Y}(s)^T \tag{4.19}
\]

Equation 4.19 corresponds to \( d \) decoupled systems of first order differential equations. In this case the Cartan matrix is independent of \( s \), so that all \( y_i(s), \ i = 1, \ldots, N \) satisfy the following differential equation with constant coefficients

\[
y'_i(s) = \mathbf{C} y_i(s) \tag{4.20}
\]
Since $\mathbb{C}$ is a skew-Hermitian matrix, it is normal and therefore has a set of $d$ independent eigenvectors $E_i$, $i = 1, \ldots, d$ forming the matrix

$$E \triangleq [E_1, \ldots, E_d] \quad (4.21)$$

Hence, Equation (4.20) has $d$ independent solutions given by

$$y_i(s) = E_i \exp \{ \lambda_i s \}, \ i = 1, \ldots, d \quad (4.22)$$

Hence, the general solution of Equation (4.19) is

$$Y = E \text{ diag}\{ \exp \{ \Lambda s \} \} \quad (4.23)$$

where $\Lambda$ is the vector of the eigenvalues of the Cartan matrix.

Consequently, the matrix of the coordinate vectors $U$ is given by

$$U = \text{ diag}\{ \exp \{ \Lambda s \} \} \ E^T \quad (4.24)$$

and the first coordinate vector $u_1(s)$ is given by

$$u_1(s) = E_1 \exp \{ \Lambda s \} \quad (4.25)$$

where $E_1$ is the first row of $E$. Hence, the natural representation of the curve can be found by integrating Equation (4.25) which yields

$$a(s) = (E_1 \odot \Lambda) \exp \{ \Lambda s \} \quad (4.26)$$

From Equation (4.26), we conclude that

$$u = E_1 \odot \Lambda \quad (4.27)$$

$$c = j \Lambda \quad (4.28)$$
Note that
\[ \| j (E_1 \otimes \Lambda) \otimes \Lambda \| = 1 \] (4.29)
and
\[ \| \Lambda \|_1 = 0 \] (4.30)
since the eigenvalues of a skew-symmetric matrix like $C$ are purely imaginary and come in conjugate pairs. In addition, none of the eigenvalues can be zero, because that would imply that the Cartan matrix would have a determinant equal to zero. However
\[ \det (C) = \kappa_1 \kappa_2 \ldots \kappa_{d-1} \]
Thus, this would signify that some of the first $d - 1$ curvatures is zero, which is contrary to the assumptions made so far.

\hfill \Box

### 4.2.1 Comments on Theorem 3

Theorem 3 states that given $d$ constant curvatures it is always possible to have a complex vector $a(s) \in \mathbb{C}^d$ which will be the natural representation of a curve represented by these curvatures. In practice, however, for array manifold curves it has been shown [11] that it is possible to have an array with $N$ elements, the array manifold curve of which will be embedded in a $u$-dimensional complex space, where $N < 2u = d$. It is useful to see how this may arise in the framework of the analysis presented in this section.

The Cartan matrix $C$ is a skew-symmetric matrix and therefore its curvatures are purely imaginary and come in conjugate pairs, which implies that
\[ \Lambda^T \Lambda = 0 \] (4.31)
Let us consider the set of $d/2$ curvatures with distinct absolute value $|\lambda_i|$. If there
is a subset of this set, with $m$ eigenvalues $\lambda_1, \ldots, \lambda_m$ such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m = 0 \quad (4.32)$$

then this implies that the $m$ corresponding entries $\exp\{\lambda_1 s\}, \ldots, \exp\{\lambda_m s\}$ are linearly dependent and are, therefore, redundant in the description of the manifold curve. Hence the dimension of the vector $a(s)$ can be reduced from $d$ to $d - m$ by removing these redundant entries.

Next two examples are going to be given, which will clarify these concepts.

**Example 1**

Let us consider the following set of curvatures.

$$\kappa = [0.5, 0.4, 0.3, 0.2, 0.1, 0]^T$$

The eigenvalues of the Cartan matrix are

$$\Lambda = j [\pm 0.6742, \pm 0.3, \pm 0.0742]^T$$

No subset of these eigenvalues adds up to 0, so the natural representation of this curve is (see Equation (4.26))

$$a(s) = \left[ \begin{array}{c} 0.4917j \\ -0.4917j \\ 0.4029j \\ -0.4029j \\ -0.3097 \\ 0.3097j \end{array} \right] \otimes \Lambda \otimes e^{\Lambda s}$$
Example 2

Let us consider the following set of curvatures.

$$\kappa = [0.6019, 0.2076, 0.2974, 0.1185, 0.2089, 0.1264, 0.1224, 0]^T$$

The eigenvalues of the Cartan matrix are

$$\Lambda = j[\pm 0.6462, \pm 0.3231, \pm 0.2261, \pm 0.0969]^T$$

Obviously, $0.3231 - 0.2261 - 0.0969 = 0$. So $m = 3$ of the eigenvalues add up to 0 and therefore the following two representations are equivalent.

$$a_1(s) = \begin{bmatrix} -j \\ -j \\ -0.7073j \\ -0.7073j \\ -0.7068j \\ -0.7068j \\ -0.7071j \\ -0.7071j \end{bmatrix} \odot \Lambda \odot \exp\{\Lambda s\}$$

and

$$a_2(s) = \begin{bmatrix} -j \\ -j \\ -0.7073j \\ -0.7068j \\ -0.7071j \end{bmatrix} \odot \begin{bmatrix} j0.6462 \\ -j0.6462 \\ -j0.3231 \\ j0.2261 \\ j0.0969j \end{bmatrix} \odot \begin{bmatrix} \exp\{j0.6462s\} \\ \exp\{-j0.6462s\} \\ \exp\{-j0.3231s\} \\ \exp\{j0.2261s\} \\ \exp\{j0.0969s\} \end{bmatrix}$$
4.3 Regular Parametric Representation of Hyperhelices in $\mathbb{C}^N$

In the previous section, Theorem 3 provided the most general form of a natural representation of a hyperhelical manifold curve lying on a complex sphere embedded in $\mathbb{C}^N$. However, the ultimate goal is to identify which array geometries can, potentially, give rise to hyperhelical spatial array manifold curves.

Using the azimuth $\theta$ and elevation $\phi$ as parameters, the spatial array manifold vector of an array of $N$ omnidirectional elements (defined in Equation (1.3)) can be written as

$$a(\theta, \phi) = \exp \left( -j\pi \begin{bmatrix} r_x, r_y, r_z \end{bmatrix} u(\theta, \phi) \right)$$  \hspace{1cm} (4.33)

where $[r_x, r_y, r_z]$ has been defined in Chapter 1 and

$$u = \begin{bmatrix} \cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi) \end{bmatrix}^T$$

The locus of the array manifold vectors as $\theta$ and $\phi$ vary, forms the array manifold surface

$$\mathcal{M} \triangleq \{ a(\theta, \phi) \in \mathbb{C}^N, \forall (\theta, \phi) : (\theta, \phi) \in \Omega_{\theta,\phi} \}$$  \hspace{1cm} (4.34)

where $\Omega_{\theta,\phi}$ is the parameter space, usually $\theta \in [0, 2\pi), \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

By enforcing a constraint

$$F(\theta, \phi) = 0$$  \hspace{1cm} (4.35)

the array manifold vector traces a curve

$$\mathcal{A} \triangleq \{ a(\theta, \phi) \in \mathbb{C}^N, \forall (\theta, \phi) : F(\theta, \phi) = 0 \land (\theta, \phi) \in \Omega_{\theta,\phi} \}$$  \hspace{1cm} (4.36)

lying on the array manifold surface of Equation (4.34). In this case, the array
manifold vector of Equation (4.33) can be rewritten as
\[
\mathbf{a}(p) = \exp \left( -j \, \mathbf{g}(p) \right)
\]
\[
\mathbf{g}(p) \triangleq \pi \left[ r_x, r_y, r_z \right] \mathbf{u}(p)
\]
provided that \( p = p(\theta, \phi) \in \Omega_p \) is an allowable change of parameter. Note that due to the constraint (4.35), \( p = p(\theta, \phi) \) is actually a function of one independent variable.

Equation (4.37) is a regular parametric representation of the spatial manifold curve \( A \) since \( p = p(\theta, \phi) \in \Omega_p \) is an allowable change of parameter. However, \( p \) is not necessarily a natural parameter and Theorem 3 cannot be readily applied. For this reason, before investigating issues related to the existence of hyperhelical array manifold curves, it is convenient to investigate what the implications of Theorem 3 are for the possible regular parametric representations of hyperhelical array manifold curves in \( \mathbb{C}^N \).

Theorem 4 presented next addresses exactly this issue, as it provides the general form of a regular parametric representation of a hyperhelical manifold curve.

**Theorem 4.** Let \( \mathcal{A} \) be the array manifold curve of an array of \( N \)-omnidirectional elements. Then, \( \mathcal{A} \) is a hyperhelix if and only if there are two constant real vectors \( \mathbf{r} \in \mathbb{R}^{N \times 1} \) and \( K \in \mathbb{R}^{N \times 1} \) and a real valued scalar function \( g(p) \) of the parameter \( p \) such, that
\[
\mathbf{a}(p) = \exp \left\{ -j \left( \mathbf{r} \, g(p) + K \right) \right\} \in \mathbb{C}^{N \times 1}
\]
is a regular parametric representation of \( \mathcal{A} \).

**Proof.** According to Theorem 3, the array manifold curve \( \mathcal{A} \) will be a hyperhelix if and only if there are two constant vectors \( \mathbf{c} \in \mathbb{R}^N \) and \( \mathbf{v} \in \mathbb{C}^N \), such that
\[
\mathbf{a}(s) = \mathbf{v} \odot \exp \left\{ -j \mathbf{c} \cdot s \right\}
\]
is a natural representation of $A$.

It was shown that any regular parametric representation for the array manifold curve $A$ can be expressed in the following format

$$a(p) = \exp\{-j\ g(p)\} \quad (4.40)$$

where $g(p) \in \mathbb{R}^N$. This implies that

$$a(p) \text{ is of class } C^1 \text{ in } I \iff g(p) \text{ is of class } C^1 \text{ in } I$$

$$\dot{a}(p) \triangleq \frac{da(p)}{dp} \neq 0, \quad \forall p \in I \iff \dot{g}(p) \neq 0, \quad \forall p \in I \quad (4.41)$$

Since expressions (4.39) and (4.40) refer to the same manifold curve, they have to be equivalent, which implies that

$$a(p(s)) = a(s) \Rightarrow$$

$$\exp\{-j\ g(p(s))\} = \exp\left\{-j(c \cdot s + \ln w - j\omega)\right\} \quad (4.42)$$

where $\ln v = \ln w - j\omega$, $v_i = |v_i|$, $-\omega_i = \arg\{v_i\}$, with $w, \omega$ constant real vectors.

Note that the general natural representation of a hyperhelical manifold curve has a complex phase or equivalently an amplitude vector $w$ different than $1_N$. However, since we are interested in array manifold curves, for Equation (4.42) to be true, we have to restrict ourselves to the case where

$$\ln w = 0_N \quad (4.43)$$

Thus, Equation (4.42) becomes

$$\exp\{-j\ g(p(s))\} = \exp\{-j(c \cdot s + \omega)\} \quad (4.44)$$

Equation (4.44) constitutes a system of $N$ equations. Let us consider the
\[ c_i s + \omega_i = g_i(p(s)) \]
\[ c_j s + \omega_j = g_j(p(s)) \]

Note that from Theorem 3 the existence of at least one \( i \) for which \( c_i \neq 0 \) is guaranteed. Equation (4.45) implies that

\[ g_j(p) = \frac{c_j}{c_i} g_i(p) + \left( \frac{\omega_j - \frac{c_j}{c_i} \omega_i}{c_i} \right) \equiv K_j \]  

(4.46)

Of course, Equation (4.46) is true \( \forall i, j = 1, \ldots, N \). Thus, \( g(p) \) can be written in the form

\[ g(p) = r g_i(p) + K \]

where \( g(p) \) can be chosen to be any of the different \( g_i(p) \), \( i = 1, \ldots, N \), for which \( c_i \neq 0 \), so that it is not constant in \( p \). Note that the conditions of Equation (4.41) guarantee that there is at least one \( g_i(p) \) such that

\[ \frac{dg_i(p)}{dp} \neq 0 \]

\[ \square \]

### 4.4 Possible hyperhelical array manifold curves

Based on the results of the previous sections it is now possible to examine which array geometries may, in principle at least, give rise to spatial array manifold curves, which can be expressed in the form of Equation (4.1). Let us write the spatial array manifold vector of an array of \( N \) omnidirectional elements as follows.

\[ \mathbf{a}(p) = \exp \left\{ -j \pi [\zeta_1, \ldots, \zeta_N]^T \mathbf{u}(p) \right\} \]  

(4.47)
where \( \mathbf{r}_i, i = 1, \ldots, N \) is the coordinates vector of the \( i \)-th array element and \( p \) is any allowable parameter, so that Equation (4.47) is a regular parametric representation of the array manifold curve. The following Theorem provides the necessary conditions for the array geometry, so that Equation (4.47) represents a hyperhelical array manifold curve in \( \mathbb{C}^N \).

**Theorem 5.** Let \( \mathcal{A} \) be the spatial manifold curve of an array of \( N \)-omnidirectional elements. Then, for \( \mathcal{A} \) to be a hyperhelix, it is necessary that the array is either linear or planar.

**Proof.** It has been proven in the previous sections that every hyperhelical manifold curve embedded in a multi-dimensional complex space \( \mathbb{C}^N \) can be expressed in the form of the natural representation (4.9). For hyperhelical manifold curves, this natural representation is equivalent to the regular parametric representation (4.38). Therefore, in order for \( \mathcal{A} \) to be of hyperhelical shape, two real vectors \( \mathbf{r}, \mathbf{K} \) have to exist, such that

\[
\exp \left\{ -j \pi \left[ \mathbf{r}_1, \ldots, \mathbf{r}_N \right]^T \mathbf{u} (p) \right\} = \exp \left\{ -j \left( \mathbf{r} \mathbf{g} (p) + \mathbf{K} \right) \right\}, \quad \forall p \in \Omega_p
\]

(4.48)

By considering two arbitrary rows \( i, j \) of the vector equation (4.48)

\[
\begin{align*}
\pi^T \mathbf{r}_i \mathbf{u} (p) &= r_i \mathbf{g} (p) + K_i \\
\pi^T \mathbf{r}_j \mathbf{u} (p) &= r_j \mathbf{g} (p) + K_j
\end{align*}
\]

(4.49)

which implies that

\[
\frac{\pi^T \mathbf{r}_i \mathbf{u} (p) - K_i}{\pi^T \mathbf{r}_j \mathbf{u} (p) - K_j} = \frac{r_i}{r_j} \Rightarrow
\]

\[
\mathbf{r}_j^T \mathbf{u} (p) = C_1 \mathbf{r}_i^T \mathbf{u} (p) + C_2, \quad \forall p \in \Omega_p
\]

(4.50)

However \( \mathbf{r}_i^T \mathbf{u} (p) \) is the norm of the projection of \( \mathbf{r}_i \) onto \( \mathbf{u} (p) \) and similarly for \( \mathbf{r}_j^T \mathbf{u} (p) \). Thus, the only cases in which Equation (4.50) holds \( \forall p \in \Omega_p \), as \( \mathbf{u}(p) \) moves in the 3-dimensional space are when
1. the unit vector $\mathbf{u}(p)$ moves in space in such a way that its component $u_{ij}$ on the plane defined by the position vectors $\mathbf{r}_i, \mathbf{r}_j$ has a constant direction, i.e. the angles between $u_{ij}$ and $\mathbf{r}_i, \mathbf{r}_j$ are independent of $p$.

2. $\mathbf{r}_i = k\mathbf{r}_j, \ k \in \mathcal{R}$

The second case clearly arises only when the array is a linear one.

The first case can only arise in the case of planar arrays and not if the array elements are placed in 3-dimensional space. To see why this is the case, let us consider a 3-dimensional array, that is an array where there are at least 3 element position vectors $\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k$ such that they are not all on the same plane. Condition (4.50) requires that the the angles between $u_{ij}$ and $\mathbf{r}_i, \mathbf{r}_j$ are constant. However, the same has to be true for the projection $u_{ik}$ of $\mathbf{u}(p)$ on the plane spanned by $\mathbf{r}_i, \mathbf{r}_k$, which is by assumption different by the plane spanned by $\mathbf{r}_i, \mathbf{r}_j$. These two restrictions, along with the assumption that $\|\mathbf{u}(p)\| = 1$ imply that $\mathbf{u}(p)$ is constant, independent of $p$.

\[\square\]

4.5 Summary

The issues of existence and uniqueness of hyperhelical manifold curves were addressed in this chapter. Initially, we proved that the natural representation of hyperhelical manifold curves embedded in $\mathbb{C}^N$ is given by Equation (4.9). This implies that all hyperhelical manifold curves lie on a complex sphere. Next, we turned our attention to hyperhelical manifold curves describing array systems. The requirement that the phase of the array manifold is real led us to restrict ourselves to a subset of all the possible hyperhelical manifold curves in $\mathbb{C}^N$, namely those for which Equation (4.43) is true. Based on this, we proved in Theorem 4 that the general parametric representation of hyperhelical array manifold curves is given by Equation (4.38). Finally, we showed in Theorem 5 that only linear and planar arrays can be described by hyperhelical array manifold curves.
Chapter 5

Array Uncertainties and Array Calibration

In most signal processing algorithms for array systems, the assumption is made that perfect knowledge of the array system is available. However, in practice, this is hardly ever true, since the exact values of the required array properties are rarely precisely known. Manufacturing errors, ageing of the array elements and environmental factors may cause the true values of the various array parameters to deviate from their nominal values. This is the problem of array uncertainties. The positions and the electrical characteristics of the array elements, the coupling matrix between them and in general any other parameter of the array system may be uncertain. These uncertainties have negative effects on the performance of the various algorithms, for example the MUSIC of an algorithm breaks down even in the case of a small uncertainty in the location of the array elements is present. This can be seen in Figure 5.1 where the performance of MUSIC has been illustrated for a perfectly calibrated array and another affected by geometric uncertainties.
There are two main ways to address the problems arising from array uncertainties. The first approach is to design array systems in such a way that they will be robust to array uncertainties. For example, one may design the geometry of an array system in such a way that small deviations from the nominal positions of the array elements have as small an effect as possible on the performance of the array system. In order to accomplish this, however, it is necessary to have an accurate modelling of the effects of array uncertainties on the performance of the array system. This is the topic of the first section of this chapter.

Some theoretical work on such error estimations has been done, providing bounds such as the Cramer-Rao bound for the estimation errors on channel parameters [44]. However, here, the goal is to derive analytical expressions which link the errors in sub-space, channel estimation algorithms to the array uncertainties. The effect of geometrical uncertainties, i.e. uncertainties in the coordinates of the array elements will be estimated for a class of sub-space based algorithms. Then, these analytical expressions can be used in the array design process.
The second approach for handling array uncertainties is array calibration. Assuming that the array characteristics are fixed, one may try to calibrate the array so that the true values of the various array parameters are discovered. The various array calibration methods can be classified as pilot or blind.

In pilot-based methods a number of sources emitting known signals are placed in known positions and measurements taken at the antenna array are used to perform array calibration. A multitude of pilot array calibration techniques have been proposed in the literature. For instance, in [47], the author uses known, time-disjoint calibration sources to estimate the positions of the array elements and a calibration matrix, which models the deviation of the true from the nominal array manifold. The problem is formulated and solved as a parameter estimation problem. In [51], again, three time-disjoint sources are employed for geometric and electrical calibration. However, in the approach, the calibration matrix is properly modelled and it is assumed to have a specific structure which facilitates the estimation of the calibration parameters. Finally, in [39], the author is using simultaneously known calibration sources to perform array geometry calibration based on the Weighted Subspace Fitting estimator.

In blind array calibration methods, the array is calibrated on-line, that is the unknown user signals are used to determine the true array characteristics. In [45], the authors utilize multiple unknown, but spectrally-disjoint far-field sources to achieve array geometry calibration and show that in any case a minimum of 3 non co-linear sources are required for calibration. Blind calibration is proposed in [61] as well, using unknown, non-disjoint sources. The algorithm achieves both array geometry calibration and DOA estimation, but again the number of required sources is greater than one. The problem of angularly dependent gain and phase uncertainties is studied in [59]. In this paper, the authors propose an auto-calibration algorithm for gain and phase uncertainties based on the assumption that a few uncertainty-free sensors exist. Finally, in [58] the author has taken advantage of the signal multipaths in order to calibrate the gain and phases of
the array elements based on the CODE criterion.

This paper focuses on array geometry calibration. In order to accurately calibrate an array system of random 3-dimensional geometry suffering from small uncertainties in the positions of the array elements, at least 3 distinct sources/paths are required, because each source/path constraints the position of each array element to be on a plane. For a co-planar array, each source gives rise to a line of potential locations, so a minimum of two sources/paths are required. The array geometry calibration problem has been sufficiently tackled in the case of multiple sources with distinct DOAs, as the aforementioned publications suggest. It is interesting, however, to examine the problem when only one source/path is available.

In the case of a single source with a fixed DOA, the positions of the array elements cannot be unambiguously determined. If, however the source is moving then, at least in principle, it is possible to derive the true array geometry. Nevertheless, the movement of the source introduces complexity in the problem, as each observation of the source corresponds to slightly different DOA and thus the performance of direction finding algorithms degrades [62], [25]. The algorithm proposed herein tackles both these issues simultaneously and achieves array geometry calibration while at the same time improving the estimation of the sources DOA.

The rest of this chapter is organised as follows. In Section 5.1 a model for predicting the induced errors in sub-space base, channel estimation algorithms is presented. In Section 5.2 a novel algorithm for array geometry calibration in the presence of a single moving source is presented.
5.1 The effect of uncertainties

5.1.1 System modelling

Consider $M$ narrow-band signals $m_i(t), i = 1, \ldots, M$ impinging on an array of $N$ omnidirectional elements. These signals can be modelled using Equation (2.9) or may be considered as realisations of zero-mean complex Gaussian random processes with power equal to $P_i$. The down-converted baseband signal at the array output $\mathbf{x}(t) \in \mathbb{C}^N$ can be modelled as (see also Equation (2.10))

$$x(t) = \sum_{i=1}^{M} S(\theta_i, \phi_i) m_i(t) + n(t) \quad (5.1)$$

where $n(t)$ is a vector of complex additive white Gaussian noise (AWGN) of power equal to $\sigma_n^2$. The vector $\mathbf{S}(\theta_i, \phi_i)$ is the spatial array manifold vector (array response vector) with the $N \times 3$ real matrix $\mathbf{r} \triangleq [r_x, r_y, r_z]$ representing the coordinates of the array elements, $(\theta_i, \phi_i)$ denoting the azimuth and elevation angles for the $i$-th incoming signal and

$$u_i = [\cos(\theta_i) \cos(\phi_i) \sin(\theta_i) \cos(\phi_i) \sin(\phi_i)]^T$$

defines the unit vector pointing towards the direction of the $(\theta_i, \phi_i)$ source. For the rest of this thesis, for the shake of simplicity and without loss of generality, only planar arrays (i.e. $r_z = 0_N$) and co-planar sources (i.e. $\phi_i = 0, \forall i$) will be considered.

Based on the aforementioned modelling, the covariance matrix $\mathbb{R}_{xx}$ of the received signals at each of the array elements can be expressed as

$$\mathbb{R}_{xx} = \mathcal{E}\left\{\mathbf{x}(t)\mathbf{x}(t)^H\right\} = \mathcal{S}\mathbb{R}_{mm}\mathcal{S}^* + \sigma_n^2\mathbb{I}_N \quad (5.2)$$

where

$$\mathcal{S} = [\mathbf{S}(\theta_1), \ldots, \mathbf{S}(\theta_M)] \in \mathbb{C}^{N \times M} \quad (5.3)$$
5.1 The effect of uncertainties

and $\mathbb{R}_{mm}$ is the covariance matrix of the signals.

The covariance matrix of Equation (5.2) is of central importance to the “subspace” type of channel estimation algorithms, such as the MUSIC algorithm, the performance of which will be investigated in this section. These algorithms assume perfect knowledge of the array geometry matrix $\mathbb{R}$. The goal of this section is to assess the accuracy of the subspace-type channel estimation methods under the presence of some uncertainty in the array geometry or the carrier frequency. However, by expressing the coordinates matrix $\mathbb{R}$ in units of half-wavelengths, the spatial manifold vector is brought in the format of Equation (1.3), so that the carrier uncertainties can be seen as a special case of geometric uncertainties.

In the following analysis, the true array geometry $\mathbb{R}$ will differ from the nominal array geometry $\mathbb{R}_o$ by some error term $\Delta \mathbb{R} = [\Delta r_x, \Delta r_y, 0_N] \triangleq \mathbb{R} - \mathbb{R}_o$, so that the true manifold vectors will be

$$\mathcal{S}(\theta_i) \triangleq \mathcal{S}(\mathbb{R}, \theta_i) = \mathcal{S}(\mathbb{R}_o + \Delta \mathbb{R}, \theta_i), \ i = 1, \ldots, M$$  \hspace{1cm} (5.4)

and the assumed true, nominal manifold vectors

$$\mathcal{S}_o(\theta_i) \triangleq \mathcal{S}(\mathbb{R}_o, \theta_i), \ i = 1, \ldots, M$$  \hspace{1cm} (5.5)

The aim is to obtain an estimate of the errors in the channel estimation (i.e. direction finding) algorithms induced by the use of the erroneous, nominal manifold vectors of Equation (5.5) instead of the correct manifold vectors of Equation (5.4).
5.1 The effect of uncertainties

5.1.2 Channel estimation errors

Single source

Initially, it is assumed that only one source is present, so that the modelling of the received signal covariance matrix \( \mathbb{R}_{xx} \) becomes

\[
\mathbb{R}_{xx} = P \mathbb{S}(\theta_1) \mathbb{S}^H(\theta_1) + \sigma_N^2 \mathbb{I}_N
\]

(5.6)

The MUSIC algorithm depends on the minimisation of the following cost function

\[
\xi(\hat{\theta}) = \mathbb{S}^H(\hat{\theta}) \mathbb{P}_n \mathbb{S}(\hat{\theta})
\]

(5.7)

where \( \mathbb{P}_n \) is the projection operation matrix on the subspace of \( \mathbb{C}^N \) spanned by the \( N - 1 \) eigenvectors corresponding to the \( N - 1 \) smallest eigenvalues of \( \mathbb{R}_{xx} \). Geometrically, the minimisation of Equation (5.7) corresponds to the intersection of the nominal array manifold curve with the \( N - 1 \) dimensional noise-subspace of \( \mathbb{C}^N \).

If the true array manifold curve is used, then MUSIC will provide the true bearing of the incoming signal \( \theta_1 \). If, however, the nominal array manifold curve is used, then an error \( \Delta \theta \) will occur. Equation (5.7) can be written as

\[
\xi(\hat{\theta}) = \mathbb{S}^H(\hat{\theta}) (\mathbb{I}_N - \mathbb{P}_s) \mathbb{S}(\hat{\theta})
\]

where \( \mathbb{P}_s \) is the projection operator matrix on the signal subspace. However, the signal subspace in this case is a 1-dimensional subspace and the normalised basis for this subspace is the normalised true array manifold vector \( \mathbb{S}(\theta_1) / \sqrt{N} \). Thus, after some algebraic manipulations, the MUSIC cost function can be written as

\[
\xi(\hat{\theta}) = N - \frac{1}{N} \left| \mathbb{S}^H(\theta_1) \mathbb{S}(\hat{\theta}) \right|^2
\]

(5.8)
5.1 The effect of uncertainties

Hence

\[ \hat{\theta}_1 = \arg\min_{\theta} \xi(\hat{\theta}) \Rightarrow \]
\[ \hat{\theta}_1 = \arg\max_{\theta} \left| S^H(\theta_1) S_o(\hat{\theta}) \right|^2 \Rightarrow \] (5.9)
\[ \hat{\theta}_1 = \arg\min_{\theta} \left\| S(\theta_1) - S_o(\hat{\theta}) \right\|^2 \]

where the last equation stems from the fact that the two manifold vectors \( S(\theta_1) \) and \( S_o(\hat{\theta}_1) \) are of equal length and relatively close to each other, because of the small error assumption.

However, because the error in the geometry is assumed to be small, \( S(\theta_1) \) can be approximated using a first order Taylor expansions around \( r_o \) (based on Equations (5.4)-(5.5)) as follows

\[ S(\theta_1) \approx S_o(\theta_1) + \frac{\partial S_o}{\partial r} \Delta r \] (5.10)

where \( r \triangleq [r_x^T, r_y^T]^T \) and \( \Delta r \triangleq [\Delta r_x^T, \Delta r_y^T]^T \). Note that

\[ \frac{\partial S_o}{\partial r} \in \mathbb{C}^{N \times 2N} \]

Similarly, expanding \( S_o(\hat{\theta}_1) \) around \( \theta_1 \) and noting that \( \hat{\theta}_1 = \theta_1 + \Delta \theta_1 \)

\[ S_o(\hat{\theta}_1) \approx S_o(\theta_1) + \frac{\partial S_o}{\partial \theta} \Delta \theta_1 \] (5.11)

Using Equations (5.10) and (5.11), Equation (5.9) becomes

\[ \Delta \hat{\theta}_1 = \arg\min_{\Delta \theta} \left\| \frac{\partial S_o}{\partial r} \Delta r - \frac{\partial S_o}{\partial \theta} \Delta \theta \right\|^2 \] (5.12)

so that the estimated error \( \Delta \hat{\theta}_1 \) of the MUSIC estimation error \( \Delta \theta_1 \) is the Least Squares solution of the overdetermined \( N \times 1 \) linear system

\[ \frac{\partial S_o}{\partial \theta} \Delta \hat{\theta}_1 = \frac{\partial S_o}{\partial r} \Delta r \]
Thus, introducing the notation $\dot{x}$ for the derivative of $x$ with respect to the azimuth $\theta$

$$\Delta \hat{\theta}_1 = \left\| \hat{S}_o^H (\theta_1) \right\|^2 \frac{\dot{S}_o^H (\theta_1)}{\partial r} \Delta r \quad (5.13)$$

Note that the term $\dot{s} \left( \underline{r}_o, \theta_1 \right) \triangleq \left\| \dot{S}_o^H (\theta_1) \right\|$ of Equation (5.13) is the rate of change of the arc length of the nominal manifold curve at the point of the true bearing $\theta_1$ [31]. The parameter $\dot{s} \left( \underline{r}_o, \theta_1 \right)$ is a measure of how fast the manifold vector “moves” as it traces the manifold curve. For a given change $\Delta \theta_1$ in the initial value $\theta_1$ of the bearing parameter $\theta$, the larger the rate of change of the arc length, the further the manifold vector $\underline{s} (\theta_1 + \Delta \theta_1)$ will be from $\underline{s} (\theta_1)$. The inverse square dependence of the estimation error on $\dot{s} \left( \underline{r}_o, \theta_1 \right)$ implies that the greater the manifold vector movement on the manifold curve as the DOA changes the smaller the estimation error. This result was expected, since a rapidly moving nominal manifold vector (i.e. large $\dot{s} \left( \underline{r}_o, \theta_1 \right)$ ) would move far away from the true vector even for a small $\Delta \theta_1$. Hence, MUSIC, searching on the nominal manifold curve for a suitable value $\Delta \theta_1$ which minimises (5.9), will produce a small estimation error.

**Multiple sources**

Assume that there are $M$ sources with bearings $\theta_i$, $i = 1, \ldots, M$. The aim is to get estimates $\hat{\theta}_i, \forall i$ of the true bearings $\theta_i, \forall i$. Following a similar reasoning as in the case of the single source scenario, the MUSIC cost function can be expressed as

$$\hat{\theta}_i = \arg \min_{\hat{\theta}} \xi \left( \hat{\theta} \right) \Rightarrow$$

$$\hat{\theta}_i = \arg \max_{\hat{\theta}} \underline{S}_o^H (\hat{\theta}) \underline{P}_s \underline{S}_o (\hat{\theta}) \quad (5.14)$$
In this case, $P_s = S (S^H S)^{-1} S^H$, where $S$ is defined in Equation (5.3). The stationary points of $\xi(\hat{\theta}) = S^H (\hat{\theta}) P_s S_o(\hat{\theta})$ can be found by setting

$$\frac{\partial \xi(\hat{\theta})}{\partial \hat{\theta}} = 0$$  (5.15)

Bearing in mind that $\hat{\theta}_i = \theta_i + \Delta \theta_i$, the nominal manifold vector $S_o(\hat{\theta}_i)$ and the first derivative of the nominal manifold vector $\dot{S}_o(\hat{\theta}_i)$ are expanded around $\theta_i$

$$S_o(\hat{\theta}_i) \simeq S_o(\theta_i) + \dot{S}_o(\theta_i) \Delta \theta_i$$

$$\dot{S}_o(\hat{\theta}_i) \simeq \dot{S}_o(\theta_i) + \ddot{S}_o(\theta_i) \Delta \theta_i$$

By using these expressions, after some algebraic manipulations, stationarity condition becomes

$$a \Delta \hat{\theta}_i^2 + b \Delta \hat{\theta}_i + c = 0$$  (5.16)

where

$$a = \text{Re}\left( \dot{S}_o^H(\theta_i) P_s \dot{S}_o(\theta_i) \right)$$

$$b = \text{Re}\left( \ddot{S}_o^H(\theta_i) P_s S_o(\theta_i) + \dot{S}_o^H(\theta_i) P_s \dot{S}_o(\theta_i) \right)$$

$$c = \text{Re}\left( \dddot{S}_o^H(\theta_i) P_s S_o(\theta_i) \right)$$  (5.17)

and again $\dddot{x}$ stands for the second derivative of $x$ with respect to $\theta$. For sufficiently small errors $\Delta \theta_i$, Equation (5.16) has 2 real solutions, the smallest of which corresponds to the estimated error $\Delta \hat{\theta}_i = \hat{\theta}_i - \theta_i$. It has to be pointed out that the same procedure can be repeated for the remaining $M - 1$ sources, in order to get the estimates $\Delta \hat{\theta}_i, \forall i$.

To complete the estimation, one has to express the projection matrix $P_s$ as a function of the nominal manifold vectors $S_o(\theta_i), \forall i$. However, the signal subspace in this case is $M$-dimensional, which means that the perturbation caused by the uncertainties, although algebraically tractable, is highly complicated. For this reason, only an outline of the procedure in the general case will be presented.
The true manifold vectors $S(\theta_1), \ldots, S(\theta_M)$ constitute a basis for the signal subspace. By applying the Gram-Schmidt procedure, an orthonormal basis $[E_1, \ldots, E_M]$ can be derived, where

$$E_1 = \frac{S(\theta_1)}{\sqrt{N}}$$

$$E_2 = \frac{S(\theta_2) - \frac{S^H(\theta_1)S(\theta_2)}{N} S(\theta_1)}{\sqrt{N - \frac{1}{N} |S^H(\theta_1)S(\theta_2)|^2}}$$

and the rest of the orthonormal basis vectors can be derived from the Gram-Schmidt procedure.

Using these orthonormal vectors, the expression for $P_s$ can be simplified as follows

$$P_s = [E_1, \ldots, E_M] [E_1, \ldots, E_M]^H$$

Noting that the new orthonormal basis vectors $E_i$, $\forall i$ actually depend on the true array geometry, they can be expanded as

$$E_i (\underline{r}) = E_i (\underline{r}_o) + \frac{\partial E_i (\underline{r}_o)}{\partial \underline{r}} \Delta \underline{r}, \ i = 1, \ldots, M$$

where

$$\underline{r} \triangleq \text{vec} \{ \underline{r} \} = [\underline{r}_x^T, \underline{r}_y^T]^T$$

$$\Delta \underline{r} \triangleq \text{vec} \{ \Delta \underline{r} \} = [\Delta \underline{r}_x^T, \Delta \underline{r}_y^T]^T$$

$$\frac{\partial E_i (\underline{r}_o)}{\partial \underline{r}} = \begin{bmatrix}
\frac{\partial E_{i,1}}{\partial r_{x,1}}, & \frac{\partial E_{i,1}}{\partial r_{x,2}}, & \cdots & \frac{\partial E_{i,1}}{\partial r_{y,1}}, & \cdots & \frac{\partial E_{i,1}}{\partial r_{y,N}} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial E_{i,N}}{\partial r_{x,1}}, & \frac{\partial E_{i,N}}{\partial r_{x,2}}, & \cdots & \frac{\partial E_{i,N}}{\partial r_{y,1}}, & \cdots & \frac{\partial E_{i,N}}{\partial r_{y,N}}
\end{bmatrix}$$

and $E_{i,1}$ is the first component of $E_i$. Substituting Equation (5.20) into Equation (5.19) the required expression in terms of the nominal geometry $\underline{r}_o$ is acquired.

In the special case where $M = 2$, which is the case in the experimental studies presented next, the required expressions can be found in Appendix E.
5.1 The effect of uncertainties

5.1.3 Effects on various array receivers

The errors in the channel parameters estimation will propagate to the weight vectors used by the receiver in order to separate and recover the signal waveforms from each source. Three different receivers will be studied, namely the Wiener-Hopf (WH) receiver, the Modified Wiener-Hopf (MWH) and a super-resolution receiver (SR).

The WH and the MWH receivers are defined as follows

\[
\mathbf{w}_{WH} = \mathbb{R}_x^{-1} \mathbf{S}_o \left( \hat{\theta}_1 \right) \tag{5.22}
\]

and

\[
\mathbf{w}_{MWH} = \mathbb{R}_{I,n}^{-1} \mathbf{S}_o \left( \hat{\theta}_1 \right) \tag{5.23}
\]

where \( \mathbb{R}_{I,n} \) is the covariance matrix of the interfering signals and the noise only. The WH receiver is optimal with regards to the Signal-to-Noise-plus-Interference (SNIR) criterion for uncorrelated sources, while the MWH is robust to pointing errors, that is errors in the estimation of the bearings of the incoming signals. This SR receiver is defined as

\[
\mathbf{w}_{SR} = \mathbb{P}_I^\perp \mathbf{S}_o \left( \hat{\theta}_1 \right) \tag{5.24}
\]

where \( \mathbb{P}_I^\perp \) is the projection operator matrix on the orthogonal subspace to the interference subspace.

In the single source scenario, the MWH reduces to

\[
\mathbf{w}_{MWH} = \frac{1}{\sigma_n^2} \mathbf{S}_o \left( \hat{\theta}_1 \right) \tag{5.25}
\]

since there are no interferences. For the same reason, the matrix \( \mathbb{P}_I^\perp \) of the SR receiver is the identity matrix and thus it is just a scaled version of the MWH.

In the multiple sources scenario, it is assumed that there are \( M \) sources and that the first one is the desired one. The presence of interferences, the power of
which will be significantly greater than the power of the noise renders the Wiener-
Hopf receiver sub-optimum. Hence, it is expected that the SR receiver, being
optimum with regards to the Signal-to-Interference (SIR) criterion will perform
better. However, since $P^\perp$ is constructed based on the estimated manifold vectors
$\Sigma_o(\hat{\theta}_2), \ldots, \Sigma_o(\hat{\theta}_M)$, there will be an error in the estimation of the interference
subspace and this will hinder its performance.

5.1.4 Experimental studies

The results of Sections 5.1.2 and 5.1.3 will be evaluated using Monte Carlo sim-
ulation studies. In all scenarios, the nominal array geometry will be a circular
array of $N$ elements, uniformly spaced on the unit circle. The errors in both the
$x$ and $y$ coordinates of the array geometry are assumed to be independent and
normally distributed with zero mean and a standard deviation $\sigma_e$ which will vary
depending on the simulation. The signal power will be assumed normalised to
unity and the power of the AWGN will be $\sigma_n^2 = 0.001$. In all simulations, the
length of the available data record was equal to 500 channel symbols and 1000
Monte Carlo simulations were run.

Single Source Case

A single source with a true direction of arrival equal to $\theta_1 = 50^\circ$ is assumed. In
Figure 5.2 the average absolute errors $\Delta \theta_1 = |\hat{\theta}_1 - \theta_1|$ are plotted, while in Figure 5.3 the % error between the predicted $\Delta \hat{\theta}_1$ using
Equation (5.13) and the $\Delta \theta_1$ of MUSIC is plotted. As expected, the average
error in the DOA estimation increases as $\sigma_e$ increases. The DOA errors are more
significant for the smaller array, because of the increased $s(\theta)$ of the larger array.

Based on Figure (5.3), the DOA errors estimation accuracy using Equation
5.1 The effect of uncertainties

(5.13) is worse for small and large values of $\sigma_e$. The former drop in accuracy is due to the very small value of $\sigma_e$, in which case the not modelled noise effects become significant compared to the effects of the uncertainties. The latter drop is due to the increase in $\Delta r$, which makes the contribution of the second and higher order terms in the expansion of Equation (5.10) significant.

In Figure 5.5 the performance of four receivers are evaluated, of which two use the true manifold vector $S(\theta_1)$ (i.e. correct array geometry and correct DOA) and two use the nominal one (i.e. the nominal array geometry) and the estimated DOA $\theta_1 + \Delta \hat{\theta}_1$. Note that the estimated DOAs are derived using MUSIC and the nominal array geometry. Note that the MWH is almost unaffected from the DOA errors as was expected. On the other hand, the performance of the WH suffers a 30-60 dB penalty due to the uncertainties.

**Multiple sources**

In this scenario, 2 sources are present, with DOAs $\theta_1 = 30^\circ$ and $\theta_2 = 70^\circ$ respectively. Figures 5.6 and 5.7 are similar to the ones presented for the channel estimation error in the single source scenario, apart from the fact that the average of the $\Delta \theta_1$ and $\Delta \theta_2$ has been plotted. The accuracy of the error prediction follows the same pattern as before.

In Figures 5.8 and 5.9 the performance of several receivers is evaluated. Again, the MWH is more robust to the DOA estimation errors. The SR receiver suffers as $\sigma_e$ increases, as expected, but still outperforms the WH. It is important to note that the modified Wiener-Hopf receiver still outperforms the conventional Wiener-Hopf and achieves almost identical performance to the super-resolution receiver.
5.1 The effect of uncertainties

Figure 5.2: MUSIC and predicted errors in the estimation of $\theta_1$

Figure 5.3: Error between the predicted and the actual estimation errors of $\theta_1$
5.1 The effect of uncertainties

Figure 5.4: SNR\textsubscript{out} for the WH and MWH receivers for an array of 6 elements.

Figure 5.5: SNR\textsubscript{out} for the WH and MWH receivers for an array of 12 elements.
5.1 The effect of uncertainties

Figure 5.6: Average MUSIC and predicted errors in the estimation of $\theta_1$ and $\theta_2$

Figure 5.7: Error between the predicted and the actual estimation errors of $\theta_1$ and $\theta_2$
5.1 The effect of uncertainties

Figure 5.8: $\text{SNIR}_{\text{out}}$, WH, MWH and SR receivers, array of 6 elements.

Figure 5.9: $\text{SNIR}_{\text{out}}$, WH, MWH and SR receivers, array of 12 elements.
5.2 Array calibration using a moving source

5.2.1 Stationary source

Consider a single, narrow-band, baseband signal \( m(t) \) impinging on a co-planar array of \( N \) omnidirectional antennae from a stationary source of direction \( \theta_0 \). It is assumed that the signal is a zero-mean complex Gaussian signal of power \( P_s \).

Let us assume, also, that there are some small uncertainties in the positions of the array elements. The true \( x \) and \( y \) sensor coordinates are given by the \( N \times 1 \) vectors \( r_x \) and \( r_y \) respectively. These differ from the nominal vectors \( r_{x,o} \) and \( r_{y,o} \) by the error vectors \( \Delta r_x \) and \( \Delta r_y \). That is \( r_x = r_{x,o} + \Delta r_x \), \( r_y = r_{y,o} + \Delta r_y \).

The baseband received signal at the antenna array \( \mathbf{x}(t) \in C^N \) can be modelled as

\[
\mathbf{x}(t) = \mathbf{S}(\theta_0) m(t) + \mathbf{n}(t)
\]

where \( \mathbf{n}(t) \) is a vector of complex additive, white Gaussian noise (AWGN) of power \( \sigma_n^2 \). The constant vector \( \mathbf{S}(\theta_0) \) is the array manifold vector (array response vector), with \( \mathbf{r} \triangleq [r_x, r_y, 0_N] \in \mathbb{R}^{N \times 3} \) and \( \mathbf{u}(\theta_0) = [\cos \theta_0, \sin \theta_0, 0]^T \) defining the unit vector pointing towards the direction \( \theta_0 \).

Let us assume that a single stationary source is observed and \( Q \) snapshots are available to the receiver. Then, the received sampled values \( \mathbf{x}(t_k) \), \( k = 1, \ldots, Q \) can be expressed as

\[
\mathbf{x}(t_k) = \mathbf{S}(\theta_0) m(t_k) + \mathbf{n}(t_k)
\]

and the sample covariance matrix \( \mathbb{R}_{xx} \) of the received baseband signal \( \mathbf{x}(t) \) can be estimated as

\[
\mathbb{R}_{xx} \approx P_s \mathbf{S}(\theta_0) \mathbf{S}(\theta_0)^H + \mathbb{R}_{nn}
\]

where \( P_s \) is the power of the signal and \( \mathbb{R}_{nn} \) is the covariance matrix of the noise of power \( \sigma_n^2 \).

The signal subspace \( \mathcal{H}_s \) of the entire observation space \( \mathcal{H} \) is spanned by the principal eigenvector of \( \mathbb{R}_{xx} \). In addition, according to Equation 5.28, the array
response vector $S(\theta_0)$ is also a basis for $\mathcal{H}_s$, i.e. $\mathcal{H}_s = \mathcal{L}[E_{xy}] = \mathcal{L}[S(\theta_0)]$. Thus

$$S(\theta_0)^H \mathbb{P}_{\mathcal{H}_s} S(\theta_0) = 0$$  \hspace{1cm} (5.29)$$

where $\mathbb{P}_{\mathcal{H}_s}$ is the projection operator matrix onto the complement of $\mathcal{H}_s$.

Equation (5.29) is the well known MUSIC cost function. However, this equation cannot be used in this case to perform array calibration, since optimization of this equation with regards to the unknown vectors $r_x$ and $r_y$ will not produce a single solution, but rather a constraint of the type $r_x \cos \theta_0 + r_y \sin \theta_0 = \text{constant vector}$. This constraint involves $2N$ unknowns in $N$ decoupled linear equations. Thus, with only one stationary source, the best that can be done is to constrain the positions of the sensors on lines perpendicular to the unknown DOA of the signal (blind) or to the known DOA of a pilot signal.

### 5.2.2 Moving source

Consider now that the source is moving with an angular speed of $\omega = \Delta \theta \times F_s \text{ rad/sec}$ where $F_s$ is the sampling frequency of the array system. The source is observed for $\Delta T = (2L + 1)T_s$ seconds and $2L + 1$ snapshots are available to the receiver. During this interval, the DOA of the moving source has changed from $\theta_0 - L\Delta \theta$ to $\theta_0 + L\Delta \theta$. Assuming that the observation interval is sufficiently small so that the angular velocity of the moving source is constant, the $2L + 1$ sampled values $\underline{x}(t_k), k = -L, \ldots, L$ can be expressed as

$$\underline{x}(t_k) = S(\theta(t_k)) m(t_k) + \underline{n}(t_k)$$

$$= \underline{S}(\theta_0 + k\Delta \theta) m(t_k) + \underline{n}(t_k)$$  \hspace{1cm} (5.30)$$

Note that this time the array response vector $\underline{S}(\theta(t_k))$ is time varying and that each snapshot is considered to be dependent on a different value of the DOA angle. For that reason the accuracy conventional DOA finding techniques such as MUSIC will decrease.
For the shake of simplicity denote $\theta_k \triangleq \theta(t_k)$, $m_k \triangleq m(t_k)$. Based on (5.30) and using a first order Taylor expansion of the $k$-th manifold vector $\hat{S}(\theta_k)$ around $\theta_0$, that is
\[
\hat{S}(\theta_k) \approx \hat{S}(\theta_0) + k \Delta \theta \hat{S} \dot{\theta}(\theta_0)
\] (5.31)
the sample covariance matrix $\mathbb{R}_{xx}$ of the received baseband signal can be written as
\[
\mathbb{R}_{xx} \approx \mathbb{SMM}^H \mathbb{S}^H + \mathbb{R}_{nn}
\] (5.32)
where $\mathbb{S} = [\hat{S}(\theta_L), \ldots, \hat{S}(\theta_L)] \in \mathbb{C}^{N \times M}$ and $\mathbb{M} = \text{diag}\{m\}$, $m = [m_{-L}, \ldots, m_L]^T$. Note that in the case of a moving source, $\mathbb{S}$ is a matrix and not a vector as in Section 5.2.1. After some algebraic manipulation Equation (5.32) can be re-written as
\[
\mathbb{R}_{xx} \approx \mathbb{P}_\psi \mathbb{S}(\theta_0) \mathbb{S}^H(\theta_0) + \mathbb{R}_{nn} + \frac{\Delta \theta^2}{2L+1} L^T \mathbb{MMM}^H L \mathbb{S}(\theta_0) \mathbb{S}(\theta_0)^H
\] (5.33)
where $L = [-L, \ldots, L]^T$.

According to Equation (5.33) the movement of the source results in the appearance of a pseudo-source with power equal to
\[
P_\psi = \frac{\Delta \theta^2}{2L+1} L^T \mathbb{MMM}^H L
\] (5.34)
The array response vector corresponding to this pseudo-source is the tangent vector $\hat{S}(\theta_0)$ to the array manifold curve at the point $\hat{S}(\theta_0)$.

The presence of this pseudo-source can be useful in the estimation of the DOA of the real moving source. Because of Equation (5.33) and the presence of the pseudo-source, it is possible to perform a search over the tangent of the array manifold curve for its intersection with the modified signal subspace consisting of the first two principal eigenvectors of $\mathbb{R}_{xx}$.

The accuracy of this DOA estimation method, which will be used in the proposed calibration algorithm, can be seen in Figure 5.10, where it is compared...
with the performance of MUSIC. Note that for this simulation both MUSIC and Tangent-MUSIC assume a 2-dimensional signal subspace. The array used is circular array of 8 sensors with half-wavelength separation. No array uncertainties were present and 10000 MC simulations were performed. For small values of the overall angular movement of the source, MUSIC outperforms the modified version since the power $P_\psi$ of the pseudo-source is minimal and its effects negligible. However, for larger speeds, the modified MUSIC outperforms the classical version.

### 5.2.3 Array calibration

The main objective of this section is to estimate $r_x$ and $r_y$, i.e. the coordinate vectors of planar arrays. The algorithm proposed is based again on Equation (5.33) and takes advantage of the presence of the pseudo-source. The signal subspace $\mathcal{H}_x$ of the entire observation space $\mathcal{H}$ is spanned by the two principal eigenvectors of $\mathbb{R}_{xx}$. In addition, the array response vectors corresponding to the
real and the pseudo-source (that is the array manifold vector $S(\mathbf{r}_x, \mathbf{r}_y, \theta_0)$ and the tangent $\dot{S}(\mathbf{r}_x, \mathbf{r}_y, \theta_0)$ to the manifold curve $\mathcal{A}$ at $\theta_0$ respectively) are also a basis for $\mathcal{H}_s$, ie $\mathcal{H}_s = \mathcal{L} \left[ S(\theta_0), \dot{S}(\theta_0) \right] = \mathcal{L} [E_1, E_2]$. This is represented by the following equations

$$S(\mathbf{r}_x, \mathbf{r}_y, \theta_0)^H \mathbb{P}_{\mathcal{H}_s} S(\mathbf{r}_x, \mathbf{r}_y, \theta_0) = 0 \quad (5.35)$$
$$\dot{S}(\mathbf{r}_x, \mathbf{r}_y, \theta_0)^H \mathbb{P}_{\mathcal{H}_s} \dot{S}(\mathbf{r}_x, \mathbf{r}_y, \theta_0) = 0 \quad (5.36)$$

where $\mathbb{P}_{\mathcal{H}_s}$ is the projection operator matrix onto the complement of the signal subspace $\mathcal{H}_s$.

It was shown in Section 5.2.1 that the positions of the array elements cannot be determined unambiguously using Equation (5.35) alone. However, Equation (5.36) results in $N$ pairs of linear equations

$$\mathbf{r} \cdot \mathbf{u}(\theta_0) = c_1 \quad \text{and} \quad \mathbf{r} \cdot \dot{\mathbf{u}}(\theta_0) = c_2 \quad (5.37)$$

the common solution of which ($N$ points) corresponds to the sensor locations. Note that $\dot{\mathbf{u}}(\theta_0)$ is the derivative of the unit vector $\mathbf{u}(\theta_0)$ with regards to the azimuth angle $\theta$. Unfortunately, although Equation (5.36) does provide a single position for each one of the array elements, its solution is very sensitive to the effects of the noise $\mathbb{R}_{nn}$. This is because $P_\psi$ is not (for reasonable velocities of the source) much larger than the power of the noise. Thus, the perturbation of the signal subspace due to the effects of the noise is significant and since $P_s \gg P_\psi$ the eigenvector corresponding to the pseudo-source is much more susceptible to the perturbations. Hence, a combined optimization method is used to derive the true coordinates vectors. The main idea behind the calibration algorithm is to take advantage of the robustness of Equation (5.35) and the unique solution of Equation (5.36) (at least in a region close to the true position of the array sensor).

The proposed algorithm is summarised below.

1. Use the nominal array geometry to get an initial estimate $\hat{\theta}_0$. 
2. Using $\hat{\theta}_0$ from Step 1, the array reference point and the known sensor, minimise the cost function of Equation (5.35) to estimate the dash-dot lines of Figure 5.11.

3. For each sensor, project the nominal sensor position on the corresponding line to get the new estimated array geometry $[\hat{r}_x, \hat{r}_y, 0_N]$.

4. Repeat Steps 1-3 using $[\hat{r}_x, \hat{r}_y, 0_N]$ as the new nominal array geometry, until the sensor positions change in Step 3 less than a specified threshold.

5. Minimize Equation (5.36) with respect to the array geometry using two constraints; the sensors must lie on the lines determined in Steps 1-4 and not exceed a predetermined distance from the nominal positions.

A representative example for the geometrical interpretation of the algorithm is given in Figure 5.11.

Steps 1-4 of the algorithm are used to determine the solution of Equation (5.35) and constrain the sensors on the specified lines (dash-dot line of Fig 5.11). In each step, the estimation of $\theta_0$ improves as well, because the projection on the constraint-lines ensures that the new estimated sensor positions will be closer to the true array geometry.

Furthermore, in the second part of the algorithm (Step 5), Equation (5.36) is optimized with regards to the array geometry. The two constraints, one acquired through Steps 1-3 and one based on reasonable assumptions about the maximum possible uncertainty in the positions of the array elements (inner square in Fig. 5.11) serve as anchors which mitigate the devastating effects of the noise in perturbing the minimum of Equation (5.36).

It has to be noted at this point that the assumption that one of the array sensors is free of errors is not indispensable for the algorithm. However, it renders the estimation of the solution of Equation (5.35) more accurate and consequently increases the final calibration accuracy.
Figure 5.11: Graphical representation of the calibration algorithm for a single array element. Note than in general there are $N - 1$ configurations like the one presented here, one for each of the sensors affected by geometric uncertainties.
5.2 Array calibration using a moving source

5.2.4 Numerical results

In this section the performance of the proposed algorithm is evaluated. Two circular arrays of 6 and 10 elements are used. In all simulations the source is assumed to have moved 6 degrees around a random DOA $\theta_0$ during the observation window and 1000 snapshots are available. The initial errors in the geometry are also random with a standard deviation of $0.1\frac{\lambda}{2}$. The optimization method used for Step 5 of the algorithm is the modified Newton method [43] and an interior-point method is implemented to model the constraints. In all simulations, 50 iterations of the optimization method were performed and no stoppage criterion was set in order to determine a lower bound for the calibration accuracy of this algorithm. It is important to note that the proposed algorithm has not been tested against other calibration algorithms, because no algorithm known to the author can perform array calibration using a single source/path. In addition, the accuracy of MUSIC and tangent-MUSIC when no calibration errors occur can be seen in Figure 5.10.

Initially, the effect of the noise is studied. In Figure 5.12 the initial and residual MSE in the positions of the array elements are plotted for different values of SNR, both for the array of 6 and the array of 10 elements. As expected, the performance of the algorithm deteriorates significantly for low values of SNR because of the severe perturbations in the signal subspace of $\mathbb{R}_{xx}$. Note that the total residual error $\mathcal{E}\left\{\|\hat{x} - \hat{e}\|_F\right\}$ in the calibration of both arrays is the same, which signifies that the residual error is mainly affected by noise and not the dimensionality of the array.

In Figure 5.13 the errors in the estimation of the true bearing $\theta_0$ are plotted, for the uncalibrated and the calibrated array. It can be seen that the improvement in the estimation of the DOA of the incoming source is significant for all values of the SNR. Finally, in Figure 5.14 the calibration improvements per iteration of the optimization method is presented. The iteration corresponds to the initial error in the positions of the array elements. The second iteration cor-
5.2 Array calibration using a moving source

responds to the estimated position of the array elements after projecting on the lines-solutions of Equation (5.35). The iterations after that correspond to the constrained optimization of Equation (5.36).

Figure 5.12: Average MSE in the positions of the array elements before and after calibration
5.2 Array calibration using a moving source

Figure 5.13: Average absolute errors in the estimation of the DOA before and after calibration

Figure 5.14: Calibration improvements per iteration of the algorithm
5.3 Summary

Array uncertainties and two different approaches for their mitigation was the main topic of this chapter. In the first part, the focus was on modelling the errors of subspace-based channel estimation techniques which were induced by geometric uncertainties in the array. Two closed form expressions were derived (Equations (5.13) and (5.16)) which can be used to design array geometries robust to the effects of uncertainties. Next, the problem of blind array calibration was tackled and a novel algorithm from geometric array calibration in the presence of a rapidly moving source was proposed.
Chapter 6

Conclusions and Future Work

The main topic of this research work has been the investigation of the newly defined extended array manifolds and the application of differential geometry in various array processing problems. In Section 6.1 a brief summary of the technical work presented in Chapters 2 - 5 is presented, while in Section 6.3 a few suggestions about possible continuation of this work are given.

6.1 Summary

In Chapter 2 the concept of the spatial array manifold has been generalised to the extended array manifold. The motivation for this generalisation is twofold. Firstly, a large number of theoretical and practical results regarding the properties of array systems exist, which are based on the geometric properties of the spatial array manifold. Nevertheless, the modelling of sophisticated array systems requires information which is not incorporated in the spatial array manifold. Therefore new array manifolds have been introduced to accomplish exactly this. The concept of the extended array manifold allows for a generic modelling of an array system, which includes virtually all the parameters of interest for the array system and encompasses many of the existing array architectures.

On the other hand, the generic concept of the extended array manifold facili-
tates the geometrical analysis of the new array manifolds. Instead of performing the same analysis every time a new array manifold is defined in the literature, generic expressions have been derived, which connect the properties of the spatial array manifold already studied in the literature to those of the extended array manifolds. Therefore, only the properties of each specific extension are required to calculate the desired geometric properties of each extended array manifold.

Having introduced the concept of the extended array manifold, the geometric properties of extended array manifold curves and surfaces have been derived as functions of the corresponding properties of the spatial manifold curves and surfaces and the properties of the associated mapping from the spatial to the extended array manifold. In the special case where the mapping producing the extended manifolds satisfies certain conditions, it has been shown that the shape of the resulting extended array manifolds remains unaltered. Hence, existing theoretical results and techniques which can be applied to only a class of manifold curves of specific geometry (i.e. hypehelices) are applicable to certain extended array manifolds as well.

Next, in Chapter 3 the problem of theoretical performance bounds for array systems has been studied. The concept of detection and resolution bounds was extended in the case of arbitrary 3-dimensional array geometries through the use of a coordinates transformation. These theoretical bounds were then carried over to the case of extended array manifolds as well, providing a theoretical tool to measure the performance of more sophisticated array systems. Finally, a new method to calculate the detection threshold has been proposed, base on the properties of the array manifold surface, which greatly simplifies the analytical and numerical calculations.

In Chapter 4, hyperhelices, a special class of array manifold curves with constant curvatures, have been studied. Hyperhelices are of special interest in array processing because their constant shape allows the closed form expression of most of their geometric properties. Moreover, due to their shape it is easier to identify
ambiguities and calculate the theoretical performance bounds which were presented in Chapter 3. Hence, an important question has been to identify which array geometries can, in principle, be described by hyperhelical manifold curves. In order to address this, first the most general natural representation of hyperhelical manifold curves embedded in $\mathbb{C}^N$ has been identified. Based on this, it was shown that only 1 and 2-dimensional array geometries can give rise to hyperhelices.

Finally, in Chapter 5 the problem of the effect of array uncertainties on the performance of array systems has been considered using an approach partially based on the concepts presented earlier in this thesis. Initially, a model of the effects of geometrical array uncertainties on channel estimation was developed and shown to accurately predict the errors present in MUSIC because of the uncertainty in the position of the array elements. Next, the problem of blind array calibration in the presence of a single moving source was investigated and a new calibration algorithm was proposed. Again, through Monte Carlo simulations it has been demonstrated that the proposed algorithm can greatly enhance the channel estimation accuracy of the array system and eliminate a large percentage of the geometrical uncertainties.
6.2 List of Contributions

The following aspects of the thesis summarise the original contributions:

1. The introduction of the concept of the extended array manifold and its definition as a complex linear mapping of the spatial array manifold in Section 2.2.

2. The investigation of the geometric properties of the extended array manifold curves and surfaces as a function of the corresponding properties of the spatial array manifold in Sections 2.3 and 2.4.

3. A theorem providing sufficient conditions for an extended array manifold curve to have a hyperhelical shape in Section 2.3.

4. A theorem linking the Gaussian curvature of spatial and hyperhelical extended array manifold surfaces in Section 2.4.

5. The extension of the detection and resolution thresholds in the case of arrays of random 3-dimensional geometry in Section 3.1.

6. The extension of the detection and resolution thresholds for any 3-dimensional array geometry, based on:
   - spatial array manifold curves.
   - extended array manifold curves.

   in Section 3.2.

7. A new approach for estimating the detection threshold based on the geometric properties of array manifold surfaces in Section 3.3.

8. A generalisation of the natural and regular parametric representations of a hyperhelical manifold curve embedded in an $N$-dimensional complex space in Sections 4.2 and 4.3.
9. The conclusion that only linear and planar arrays may be described by manifold curves of constant curvatures in Section 4.4.

10. The modelling of the effects of geometrical and carrier uncertainties on the performance of subspace-based channel estimation methods in Section 5.1.

11. An algorithm for mitigating the effects of geometrical/carrier uncertainties and movement for an array system in the presence of a single moving source in Section 5.2.

6.3 Future Work

The aim of this research work has been to investigate the properties of the extended array manifolds and especially the nature of their relations with the widely studied spatial array manifold. Although this investigation managed to clarify many issues regarding the geometry of the extended array manifolds, their study is far from complete. It would be interesting to examine the geometry of delay manifold curves, that is curves which are parameterised using the lack of synchronisation of the incoming signal. This is a relatively challenging task, since in the existing modelling the delay is considered to be an integer and not a continuous quantity. Another interesting task, would be the investigation of a possible relation between the shape of the manifold curves and the ambiguities an array system presents. In the case of constant shape curve, namely hyperhelices, a large number of ambiguities have been identified. However, for non-constant curvature curves this task has not yet been accomplished.

Furthermore, in Chapter 3 a new approach was presented to estimate the detection threshold for any given array system based on the differential geometric properties of the manifold surface. However, this has not been accomplished for the resolution threshold. An extension of this surface-based approach to the resolution threshold will probably result in dramatic decrease in the computational burden of calculating this theoretical bound.
A significant part of the technical work presented in this thesis could be used in the array design problem. The theoretical performance bounds, as they were presented in Chapter 3 can be used as a design criterion for array systems. Another design criterion could be the robustness of array systems to the effects of geometrical uncertainties, which were modelled in Chapter 5. These criteria could be used in concert with previously proposed design criteria, such as the sensitivity of the array system with regards to individual array sensors [49], [1].

Finally, array calibration seems to be a not yet fully resolved issue in array processing. Especially the case where the sources are not static has received virtually no attention in the literature, despite the fact that scenarios with moving sources are encountered frequently in array processing. The case where multiple moving sources are available is a plausible and challenging continuation of the work presented herein. The existence of more than one source is likely to render the problem solvable with greater accuracy due to the reasons explained in Section 5.2. However, the significantly increased degrees of freedom inherent in the problem also result in an increase in the complexity of the problem.
Appendix A

Proof of Theorem I

The general formulae for calculating the $i$-th coordinate vector and curvature can be found in [30,31].

A.1 $i$-th coordinate vector and curvature

Based on the results for the first and second order coordinate vectors and curvatures, the emerging pattern of the formulae leads to the following expressions:

$$u_i = \frac{\sigma^j_i}{\sigma^j_i \kappa_1 \kappa_2 \ldots \kappa_{i-1}} A \left[ \left( \begin{array}{c} [\frac{i-1}{2}] +1 \\ \sum_{n=1}^{[\frac{i}{2}]+1} (-1)^{n-1} b_{i-1,n} \mathcal{L}^{i-2n+2} \end{array} \right) \odot a \right]$$  \hspace{1cm} (A.1)

$$\kappa_i = \frac{\sigma^{-1}_i}{\kappa_1 \kappa_2 \ldots \kappa_{i-1}} \left\| \left( \begin{array}{c} [\frac{i}{2}] +1 \\ \sum_{n=1}^{[\frac{i}{2}]+1} (-1)^{n-1} b_{i,n} \mathcal{L}^{i-2n+3} \end{array} \right) \right\|$$  \hspace{1cm} (A.2)

where

$$b_{i,n} = b_{i-1,n} + \sigma^2_i \kappa^2_{i-1} b_{i-2,n-1} \quad , \quad i \geq 1$$  \hspace{1cm} (A.3)
These expressions will now be proven.

Proof. We will begin with the proof of Equation (A.1) and the method of 2-step mathematical induction will be used to prove the desired expression. Note that Equation (A.1) is valid for \( i \in \mathbb{Z}, i \geq 1 \).

\( i = 1 \) and \( i = 2 \)

If we apply formula (A.1) for \( i = 2 \) and \( i = 3 \) it is straightforward to show that it is verified.

\( i = k - 1 \), \( i = k \)

Let us assume that this formula is correct for \( i = k - 1 \) and \( i = k \), that is

\[
\begin{align*}
{u}_{k-1} &= \frac{j^{k-1}}{\sigma_k^{k-1} \kappa_1 \kappa_2 \cdots \kappa_{k-2}} \hat{A} \left( \left( \sum_{n=1}^{\lfloor \frac{k-2}{2} \rfloor +1} (-1)^{n-1} \cdot {b}_{k-2,n} \hat{L}_n^k \right) \odot \hat{a} \right)
\end{align*}
\]

and

\[
\begin{align*}
{u}_k &= \frac{j^k}{\sigma_k^{k-1} \kappa_1 \kappa_2 \cdots \kappa_{k-1}} \hat{A} \left( \left( \sum_{n=1}^{\lfloor \frac{k-1}{2} \rfloor +1} (-1)^{n-1} \cdot {b}_{k-1,n} \hat{L}_n^{k+2-2n} \right) \odot \hat{a} \right)
\end{align*}
\]
Then

$$u'_k = \frac{j^{k+1}}{\sigma_{\mathcal{A}}^{k+1} \kappa_1 \kappa_2 \cdots \kappa_{k-1}} A \left[ \left( \sum_{n=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor +1} (-1)^{n-1} \cdot b_{k-1,n} L^{k+3-2n} \right) \odot a \right]$$

and

$$\kappa_{k-1} u_{k-1} = \frac{j^{k-1} \kappa_k^2}{\sigma_{\mathcal{A}}^{k-1} \kappa_1 \kappa_2 \cdots \kappa_{k-1}} A \left[ \left( \sum_{n=1}^{\left\lfloor \frac{k-2}{2} \right\rfloor +1} (-1)^{n-1} b_{k-2,n} L^{k+3-2n} \right) \odot a \right]$$

By setting $m \triangleq n + 1$

$$\kappa_{k-1} u_{k-1} = \frac{-j^{k+1} \sigma_{\mathcal{A}}^{k+1} \kappa_k^2}{\sigma_{\mathcal{A}}^{k+1} \kappa_1 \kappa_2 \cdots \kappa_{k-1}} A \left[ \left( \sum_{m=2}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{m-2} b_{k-2,m} L^{k+3-2m} \right) \odot a \right]$$

Substituting again $n \triangleq m$, we get

$$k_{k-1} u_{k-1} = \frac{j^{k+1} \sigma_{\mathcal{A}}^{k+1} \kappa_k^2}{\sigma_{\mathcal{A}}^{k+1} \kappa_1 \kappa_2 \cdots \kappa_{k-1}} A \left[ \left( \sum_{n=2}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{n-1} b_{k-2,n-1} L^{k+3-2n} \right) \odot a \right].$$
\( i = k + 1 \)

\[
\frac{u_{k+1}}{\kappa_k} = \frac{u'_k + \kappa_{k-1} u_{k-1}}{\kappa_k} =
\]

\[
\frac{j^{k+1}}{\sigma_{k_1}^{k+1} \kappa_1 \kappa_2 \cdots \kappa_k} \mathbb{A}^k \left( \sum_{n=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^{n-1} b_{k-1,n} \tilde{r}_k^{k+3-2n} \right) \odot \mathbb{A} \)  

(A.6)

The formula of Equation (A.6) is not the same as the one we are trying to prove. From the first sum of the right hand side, the term for \( n = \left\lfloor \frac{k}{2} \right\rfloor + 1 \) is missing, while from the second sum, the term for \( n = 1 \) is missing. We will show now, that both these terms equal to 0, or equivalently, that the corresponding coefficients \( b_{i,n} = 0 \).

The coefficients \( b_{i,n} \) are 0 if

1. \( i = n \land i > 2 \)
2. \( i < n \)
3. \( n = 0 \)

Therefore, the second missing term can easily be proven to be equal to 0, since

\[
b_{k-2,n-1} = b_{k-2,0} = 0
\]

If \( k = 2 \rho + 1 \), \( \rho = 1, 2 \ldots \), then \( \left\lceil \frac{k}{2} \right\rceil + 1 = \rho + 1 = \left\lceil \frac{k-1}{2} \right\rceil + 1 \). Therefore, there is no need to show that the term under consideration is equal to 0.

If \( k = 2 \rho \), \( \rho = 1, 2 \ldots \) then the term under consideration is

\[
n = \left\lfloor \frac{k}{2} \right\rfloor + 1 = \rho + 1 : (-1)^{\rho} \cdot b_{2 \rho-1, \rho+1} \cdot \tilde{r}
\]
Therefore, we have to show that $b_{2p-1,p+1} = 0$.

The coefficients $b_{i,n}$ can be defined recursively, as can be seen in Equation (A.3). In every step of the recursion, the coefficient can be expressed as the sum of two terms, in each one of which there is another coefficient, with different indices. In every step, in both of the emerging coefficients, the difference between the first and the second index is decreased by one. However, in the second term, the first index is decreased each time by two and the second by one, while in the first term only the first index is decreased by one. In order for $b_{2p-1,p+1} = 0$ to be zero, all of the emerging coefficients in this recursion must be also zero. Equivalently, the indices of all the emerging coefficients must satisfy one of the conditions mentioned above. We can represent this recursion as a binary tree, see Figure A.1.

![Figure A.1: Graphical representation of the iterative generation of $b_{i,n}$](image)

Figure A.1, each node of which will represent a different coefficient. For the root of this tree to be 0, all the leafs have to be zero too. However, if we prove that the coefficient emerging always from the second term of the recursive equation (A.3), that is the rightmost leaf of the tree is equal to 0 then, all the other coefficients will be equal to 0 as well, because even though the difference in the indexes is decreased by the same amount in each recursion (each level of the tree), the indices of the rightmost coefficients (the nodes in the path from the root to the
rightmost coefficient) approach the limit $i = 2$ faster than the nodes in any other path.

After $x$ steps of the recursion, the indices of the rightmost leaf will be

$$b_{i_x,n_x} = b_{2 \rho - 1 - 2x, \rho + 1 - x}$$

The indices have to be equal, in order for the coefficient to be zero, so

$$2 \rho - 1 - 2x = \rho + 1 - x \Rightarrow x = \rho - 2$$

Therefore, after $x = \rho - 2$ steps of the recursion, indices will be

$$i_x = 2 \rho - 1 - 2 (\rho - 2) = 3 > 2$$
$$n_x = \rho + 1 - (\rho - 2) = 3 > 2$$

Thus, it has been proven, that all the coefficients for this term are 0 and the proof for the coordinate vectors is complete. The proof for the curvatures is straightforward using Equation (A.2), since this formula is based on the formula of the coordinate vectors.

\[ \square \]

A.2 Proof of Theorem 1

Proof.

The method of mathematical induction will be used again. During this proof, variables $b_{RE}$ and $\kappa_{RE}$ will refer to the coefficients and the curvatures of the extended manifold curves, while variables $b$ and $\kappa$ will refer to the coefficients and curvatures of the spatial manifold curves.

Based on Eq. (A.2), the calculation of the the $i$-th curvature $\kappa_i$ requires the knowledge of all the coefficients $b_{k,n}$, $k = 1, \ldots, i$, $n = 1, \ldots, \left\lfloor \frac{i}{2} \right\rfloor$, whereas
the calculation of the coefficients $b_{i,n}, n = 1, \ldots, \left\lfloor \frac{i}{2} \right\rfloor$ require the knowledge of the curvature $\kappa_{i-1}$ and the that of some of the previous coefficients. Therefore, the proof of the two equations has to be simultaneous.

Let us write down the necessary coefficients for the calculation of the curvature $\kappa_i$ of the spatial manifold.

$$
\begin{pmatrix}
  b_{1,1} \\
  b_{2,1} & b_{2,2} \\
  \vdots & \vdots & \ddots \\
  b_{i,1} & b_{i,2} & \ldots & b_{i,\left\lfloor \frac{i}{2} \right\rfloor}
\end{pmatrix}
$$

Based on this arrangement and Equation (A.3) the calculation of each coefficient is a function of the coefficients which precede it in the previous arrangement plus the curvature of the previous order. Therefore, if it is known that up to an order $i$

$$
\begin{align*}
  b_{k,n}^{H_c} &= b_{k,n}, \quad k = 1, \ldots, i, \quad n = 1, \ldots, \left\lfloor \frac{i}{2} \right\rfloor \\
  \kappa_k^{H_c} &= \kappa_k \sigma_k, \quad k = 1, \ldots, \left\lfloor \frac{i}{2} \right\rfloor
\end{align*} \quad (A.7)
$$

then for the next order $i + 1$, the coefficients of the spatial manifold and the extended manifold will be equal as well. Since Equation (A.3) is a two step recursion, we have to prove first that the necessary conditions hold for $i = 1$ and $i = 2$.

For $i = 1$, it is

$$
\begin{align*}
  b_{1,n}^{H_c} &= [1] = b_1 \\
  \kappa_1^{H_c} &= \frac{\|\tilde{r}_2\|}{\sigma_k} = \frac{\kappa_1}{\sigma_k}
\end{align*}
$$
For $i = 2$, it is

$$b_{2,2}^H = \sigma_\kappa^2 \left( \kappa_1^H \right)^2 = \kappa_1^2 = b_{2,2}$$

$$\kappa_2^H = \frac{1}{\sigma_\kappa^2 \kappa_1^H} \left\| \bar{\partial}^3 - \sigma_\kappa^2 \left( \kappa_1^H \right)^2 \bar{\partial} \right\| = \frac{\kappa_2}{\sigma_\kappa}$$

Now, it has to be proven that, if the conditions of Equation (A.7) hold and given the previous result for the coefficients, the curvatures relation of Equation (A.7) will hold for the next order $i + 1$ as well.

For the next order, $i + 1$, based on Equation (A.7), the coefficients $b_{i+1}^H$ and $b_i$ will be equal. Thus, from Equation (A.2) it is straightforward to show that

$$\kappa_{i+1}^H = \frac{\kappa_{i+1}}{\sigma_\kappa} \quad \text{(A.8)}$$

Now that the curvatures relation has been proven, by substituting Equation (A.8) into Equation (A.1) it is straightforward to show that Equation (A.1) is also verified.
Appendix B

Derivation of Christoffel Matrices

B.1 Derivation of Equation (2.71)

Based on the definitions of the Christoffel matrix of the first kind

\[ \Gamma_{1,\mathcal{H}} = \text{Re}\left\{ \hat{T}^H_{\zeta,\mathcal{H}} \hat{T}_{\mathcal{H}} \right\} \]  

(B.1)

However

\[ \hat{T}_{\zeta,\mathcal{H}} = \dot{A}_{pq} I_2 \otimes \dot{a}_\zeta + \ddot{A}_{pq,\zeta} I_2 \otimes a + \dot{A}_\zeta \mathcal{T} + A \hat{T}_\zeta \]

where

\[ \ddot{A}_{pq,\zeta} \triangleq \frac{\partial \dot{A}_{pq}}{\partial \zeta} \quad \text{and} \quad \dot{T}_\zeta \triangleq \frac{\partial \mathcal{T}}{\partial \zeta} \]
Thus

\[
\text{Re} \left\{ T_{\dot{\zeta}}^{\mu} \right\} = \text{Re} \left\{ \left( \dot{A}_{pq} \mathbb{I}_2 \otimes \dot{\alpha}_c + \ddot{A}_{pq,\zeta} \mathbb{I}_2 \otimes \alpha + \dot{\alpha}_c^T + A \dot{T}_\zeta \right)^H \left( \dot{A}_{pq} \mathbb{I}_2 \otimes \alpha + A \mathbb{T} \right) \right\} 
\]

\[
= \text{Re} \left\{ \left( \mathbb{I}_2 \otimes \dot{\alpha}_c^H \dot{A}_{pq}^H \dot{\alpha}_c \mathbb{I}_2 \otimes \alpha + \mathbb{I}_2 \otimes \dot{\alpha}_c^H \dot{A}_{pq}^H A \mathbb{T} + \mathbb{I}_2 \otimes \alpha \dot{A}_{pq}^H \dot{\alpha}_c \mathbb{I}_2 \otimes \alpha \right) \right\} 
\]

\[
+ \text{Re} \left\{ \mathbb{I}_2 \otimes \ddot{A}_{pq,\zeta}^H \dot{\alpha}_c \mathbb{I}_2 \otimes \alpha + \mathbb{T} \dot{A}_{pq,\zeta}^H \dot{\alpha}_c \mathbb{I}_2 \otimes \alpha + \mathbb{T} \dot{A}_{pq,\zeta}^H A \mathbb{T} \right\} 
\]

\[
+ \text{Re} \left\{ T_{\dot{\zeta}}^H \dot{\alpha}_c^H \mathbb{I}_2 \otimes \alpha + T_{\dot{\zeta}}^H \dot{\alpha}_c^H A \mathbb{T} \right\} 
\]

(B.2)

Terms 1, 4 and 5 are all equal to \(\mathbb{O}_2\) since they involve the inner products \(\dot{\alpha}_c^H \alpha = 0\).

Term 2:

\[
\text{Re} \left\{ \mathbb{I}_2 \otimes \dot{\alpha}_c^H \dot{A}_{pq}^H A \mathbb{T} \right\} = \text{Re} \left\{ \left[ \begin{array}{cc} \dot{\alpha}_c^H \dot{A}_{pq}^H \dot{\alpha}_c & \dot{\alpha}_c^H \dot{A}_{pq}^H \dot{\alpha}_c \\ \dot{\alpha}_c^H \dot{A}_{pq}^H \dot{\alpha}_c & \dot{\alpha}_c^H \dot{A}_{pq}^H \dot{\alpha}_c \\ \dot{\alpha}_c^H \dot{\alpha}_c & \dot{\alpha}_c^H \dot{\alpha}_c \\ \dot{\alpha}_c^H \dot{\alpha}_c & \dot{\alpha}_c^H \dot{\alpha}_c \end{array} \right] \otimes \left[ \begin{array}{cc} \dot{z}_p^H \dot{\bar{z}}_p & \dot{z}_p^H \dot{\bar{z}}_p \\ \dot{z}_q^H \dot{\bar{z}}_q & \dot{z}_q^H \dot{\bar{z}}_q \end{array} \right] \right\} 
\]

\[
= \text{Re} \left\{ T_{\dot{\zeta}}^H \dot{\alpha}_c^H \mathbb{T} \right\} 
\]

(B.3)
Term 3:

\[
\text{Re} \left\{ \mathbb{I}_2 \otimes \hat{a}^H \hat{a}_{pq \zeta} \hat{A}_{pq} \mathbb{I}_2 \otimes \hat{a} \right\} = \text{Re} \left\{ \begin{bmatrix} \hat{a}^H \hat{A}_{pq \zeta} \hat{A}_{pq} \hat{a} & \hat{a}^H \hat{A}_{pq \zeta} \hat{A}_{pq} \hat{a} \end{bmatrix} \right\}
\]

\[
= \text{NRe} \left\{ \begin{bmatrix} \hat{z}^H \hat{z}_{pq \zeta} \hat{z}_{pq \zeta} \hat{z} \end{bmatrix} \right\}
\]

\[
= \text{NRe} \left\{ \hat{T}^H_{\hat{z}_\zeta} - \hat{T}_{\hat{z}_\zeta} \right\}
\]

Term 6:

\[
\text{Re} \left\{ \mathbb{I}_2 \otimes \hat{a}^H \hat{A}_{pq} \hat{A}_T \right\} = \text{Re} \left\{ \begin{bmatrix} \hat{a}^H \hat{A}_{pq} \hat{A}_T \hat{a} & \hat{a}^H \hat{A}_{pq} \hat{A}_T \hat{a} \end{bmatrix} \right\}
\]

\[
= \text{Re} \left\{ \begin{bmatrix} \hat{z}^H \hat{A}_{pq} \hat{A}_T \hat{z} & \hat{z}^H \hat{A}_{pq} \hat{A}_T \hat{z} \\
\hat{z}^H \hat{A}_{pq} \hat{A}_T \hat{z} & \hat{z}^H \hat{A}_{pq} \hat{A}_T \hat{z} \end{bmatrix} \right\}
\]

\[
= \text{Re} \left\{ \hat{z}^H \hat{z} - \hat{z}^H \hat{G} \hat{z} \right\}
\]

Term 7:

\[
\text{Re} \left\{ \hat{T}^H_{\hat{z}_\zeta} \hat{A}_{pq} \hat{A}_T \mathbb{I}_2 \otimes \hat{a} \right\} = \text{Re} \left\{ \begin{bmatrix} \hat{z}^H \hat{A}_{pq} \hat{A}_T \hat{a} & \hat{z}^H \hat{A}_{pq} \hat{A}_T \hat{a} \end{bmatrix} \right\}
\]

\[
= \text{Re} \left\{ \begin{bmatrix} \hat{z}^H \hat{A}_{pq} \hat{a} & \hat{z}^H \hat{A}_{pq} \hat{a} \\
\hat{z}^H \hat{A}_{pq} \hat{a} & \hat{z}^H \hat{A}_{pq} \hat{a} \end{bmatrix} \right\} \otimes \left\{ \begin{bmatrix} \hat{z} \hat{z}_p & \hat{z} \hat{z}_q \end{bmatrix} \right\}
\]

\[
= \text{Re} \left\{ \hat{T}^H_{\hat{z}_\zeta} \hat{a} - \hat{z}^H \hat{T}_{\hat{z}_\zeta} \hat{z} \right\}
\]
B.2 Derivation of Equation (2.76)

Initially the inverse of the manifold metric $\mathcal{G}_H$ of the extended manifold surface has to be derived.

\[
\mathcal{G}_H^{-1} = \frac{1}{\det \mathcal{G}_H} \begin{bmatrix}
N \dot{z}_q \dot{z}_q + \sigma^2 \dot{z}_p \dot{z}_p & \text{Re} \left\{ -N \dot{z}_q \dot{z}_p - \sigma^2 \dot{z}_p \dot{z}_p \right\} \\
\text{Re} \left\{ -N \dot{z}_q \dot{z}_p - \sigma^2 \dot{z}_p \dot{z}_p \right\} & N \dot{z}_p \dot{z}_p + \sigma^2 \dot{z}_q \dot{z}_q \\
\end{bmatrix}
\]

\[
= \frac{\sigma^2 \mathcal{G} \mathcal{G}_H^{-1} + N \mathcal{G}_z \mathcal{G}_z^{-1}}{\det \mathcal{G}_H}
\]

However

\[
\det \mathcal{G}_H = N^2 \left( \dot{z}_p \dot{z}_q - \text{Re} \left\{ \dot{z}_p \dot{z}_q \right\} \right) \\
+ \sigma_0 \left( \dot{z}_p \dot{z}_q + \sigma_0 \dot{z}_p \dot{z}_q \right) \text{Re} \left\{ \dot{z}_p \dot{z}_q \right\} \\
+ N \sigma_0^2 \left\{ \dot{z}_p \dot{z}_q \text{Re} \left\{ \dot{z}_p \dot{z}_q \right\} - \text{Re} \left\{ \dot{z}_p \dot{z}_q \right\} \text{Re} \left\{ \dot{z}_p \dot{z}_q \right\} \right\} \\
+ \text{Tr} \{ \det \mathcal{G}_z \mathcal{G}_z^{-1} \} \\
= N^2 \det \mathcal{G}_z + \sigma_0 \det \mathcal{G}_z + N \sigma_0^2 \text{Tr} \{ \det \mathcal{G}_z \mathcal{G}_z^{-1} \}
\]
Appendix C

Derivation of Eq. 3.11

Initially, the first coordinate vector has to be calculated.

\[ \mathbf{u}_1(\alpha) = \frac{\partial \mathbf{a}}{\partial \alpha} \cdot \frac{d\alpha}{ds} \quad (C.1) \]

It is

\[ \dot{\mathbf{a}}(\alpha) \triangleq \frac{\partial \mathbf{a}}{\partial \alpha} = j \pi \sin \alpha \mathbf{w}(\alpha) \]

and

\[ \|\dot{\mathbf{a}}(\alpha)\| = \pi \sin \alpha \|\mathbf{w}(\alpha)\| \quad (C.2) \]

where

\[ \mathbf{w}(\alpha) = \mathbf{r}(\Theta_o) - \frac{\cos \alpha}{\sqrt{1 - \cos^2(\alpha) - \cos^2(\beta_o) f(\alpha)}} r_z \]
\[ \|w(\alpha)\| = \sqrt{(r(\Theta_o) - f(\alpha) \cdot r_z)^T (r(\Theta_o) - f(\alpha) \cdot r_z)} \]

Thus
\[ u_1(\alpha) = \frac{\dot{\alpha}(\alpha)}{\|\dot{\alpha}(\alpha)\|} = j \dot{w}(\alpha) \odot \dot{a}(\alpha) \]  

(C.3)

where
\[ \dot{w}(\alpha) \triangleq \frac{w(\alpha)}{\|w(\alpha)\|} \]  

(C.4)

Differentiating again
\[ u_1'(\alpha) = \frac{du_1}{ds} = j (\dot{w}(\alpha) \odot \ddot{a}(\alpha) + \ddot{w}(\alpha) \odot \dot{a}(\alpha)) \frac{1}{\|\dot{a}(\alpha)\|} \]  

(C.5)

After some algebra
\[ u_1'(\alpha) = \left[ -\ddot{w}^2(\alpha) + \frac{j}{\pi \|w(\alpha)\|} (G(\alpha) r(\Theta_o) - H(\alpha) r_z) \right] \odot \dot{a}(\alpha) \]  

(C.6)

where
\[ G(\alpha) = \frac{\sin \alpha}{g(\alpha)} \left( \frac{\|r_z\|^2 \cos \alpha}{g(\alpha)} - r(\Theta_o)^T r_z \right) \left( 1 + \frac{\cos^2 \alpha}{g'(\alpha)} \right) \]  

(C.7)

and
\[ H(\alpha) = -\frac{\sin \alpha}{g(\alpha)} \left( \frac{1 + \cos^2 \alpha}{g'(\alpha)} \right) \frac{\cos \alpha}{g(\alpha)} G(\alpha) \]  

(C.8)

Based on Equations (C.6) - (C.8) the first curvature is
\[ \kappa_1(\alpha) = \left\| -\ddot{w}^2(\alpha) + \frac{j}{\pi \|w(\alpha)\|} (G(\alpha) \cdot r(\Theta_o) - H(\alpha) \cdot r_z) \right\| \]  

(C.9)
Figure C.1: Principal curvature of the different $\alpha$-curves when $\phi = 90^\circ$ for the array of Figure 3.2
Appendix D

Proof of the curvatures formula of Theorem 3

We will begin with the proof of Equation (D.1) and the method of 2-step mathematical induction will be used to prove the desired expression. Note that the approach is similar with the one in Appendix A, however it is given here for the shake of completeness.

Proof. Based on the results for the first and second order coordinate vectors and curvatures, the emerging pattern of the formulae leads to the the following expression for the $i$-th coordinate vector:

$$u_i = \frac{j^i}{\kappa_1 \kappa_2 \ldots \kappa_{i-1}} \cdot \left[ \left( \sum_{n=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor + 1} (-1)^{n-1} \ b_{i-1,n} \ e^{i-2n+2} \right) \odot a \right] \quad (D.1)$$

Equation (D.1) is valid for $i \in \mathbb{Z}, i \geq 2$.

$i = 2$ and $i = 3$

If we apply formula (D.1) for $i = 2$ and $i = 3$ it is straightforward to show that it is verified.
\( i = k - 1, \ i = k \)

Let us assume that this formula is correct for \( i = k - 1 \) and \( i = k \), that is

\[
\begin{align*}
\kappa_{k-1} \cdot u_{k-1} &= \frac{j^{k-1}}{\kappa_1 \kappa_2 \cdots \kappa_{k-2}} \cdot \left[ \left( \sum_{n=1}^{\left\lfloor \frac{k-2}{2} \right\rfloor +1} (-1)^{n-1} \cdot b_{k-2,n} \cdot \zeta^{k+1-2n} \right) \right] \circ a \\
\kappa_k \cdot u_k &= \frac{j^k}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{n=1}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{n-1} \cdot b_{k-1,n} \cdot \zeta^{k+2-2n} \right) \right] \circ a
\end{align*}
\]

Then

\[
\begin{align*}
\kappa_{k-1} \cdot u'_{k-1} &= \frac{j^{k+1}}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{n=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor +1} (-1)^{n-1} \cdot b_{k-2,n} \cdot \zeta^{k+3-2n} \right) \right] \circ a \\
\kappa_k \cdot u'_k &= \frac{j^{k+1}}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{n=1}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{n-1} \cdot b_{k-1,n} \cdot \zeta^{k+3-2n} \right) \right] \circ a
\end{align*}
\]

By setting \( m \triangleq n + 1 \)

\[
\begin{align*}
\kappa_{k-1} \cdot u_{k-1} &= \frac{-j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{m=2}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{m-2} b_{k-2,m-1} \cdot \zeta^{k+3-2m} \right) \right] \circ a \\
\kappa_k \cdot u_k &= \frac{-j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{m=2}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{m-2} b_{k-2,m-1} \cdot \zeta^{k+3-2m} \right) \right] \circ a
\end{align*}
\]

Substituting again \( n \triangleq m \), we get

\[
\begin{align*}
\kappa_{k-1} \cdot u_{k-1} &= \frac{j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{n=2}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{n-1} b_{k-2,n-1} \cdot \zeta^{k+3-2n} \right) \right] \circ a \\
\kappa_k \cdot u_k &= \frac{j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdot \kappa_2 \cdots \kappa_{k-1}} \cdot \left[ \left( \sum_{n=2}^{\left\lfloor \frac{k}{2} \right\rfloor +1} (-1)^{n-1} b_{k-2,n-1} \cdot \zeta^{k+3-2n} \right) \right] \circ a
\end{align*}
\]
\[ i = k + 1 \]

\[
\frac{u_{k+1}}{u_k} = \frac{u'_k + \kappa_{k-1} \cdot u_{k-1}}{\kappa_k} = \frac{j^{k+1}}{\kappa_1 \cdot \kappa_2 \cdots \kappa_k} \cdot \left[ \left( \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \sum_{n=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (-1)^{n-1} b_{k-1,n} \cdot \xi^{k+3-2n} \right) \odot a \right] + \frac{j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdot \kappa_2 \cdots \kappa_k} \cdot \left[ \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \sum_{n=2}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^{n-1} b_{k-2,n-1} \cdot \xi^{k+3-2n} \right) \odot a \right] \tag{D.2}
\]

The formula of Equation (D.2) is not the same as the one we are trying to prove. From the first sum of the right hand, the term for \( n = \left\lfloor \frac{k}{2} \right\rfloor + 1 \) is missing, while from the second sum, the term for \( n = 1 \) is missing. We will show now, that both these terms equal to 0, or equivalently, that the corresponding coefficients \( b_{i,n} = 0 \).

The coefficients \( b_{i,n} \) are 0 if

1. \( i = n \land i > 2 \)
2. \( i < n \)
3. \( n = 0 \)

Therefore, the second missing term can easily be proven to be equal to 0, since

\[ b_{k-2,n-1}^{n=1} = b_{k-2,0} = 0 \]

If \( k = 2\rho + 1 \), \( \rho = 1, 2 \ldots \), then \( \left\lfloor \frac{k}{2} \right\rfloor + 1 = \rho + 1 = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 \). Therefore, there is no need to show that the term under consideration is equal to 0.
If $k = 2\rho$, $\rho = 1, 2 \ldots$ then the term under consideration is

$$n = \left\lfloor \frac{k}{2} \right\rfloor + 1 = \rho + 1 : (-1)^{\rho} \cdot b_{2\rho - 1, \rho + 1} \cdot c$$

Therefore, we have to show that $b_{2\rho - 1, \rho + 1} = 0$.

The coefficients $b_{i,n}$ can be defined recursively, as can be seen in Equation (4.13). In every step of the recursion, the coefficient can be expressed as the sum of two terms, in each one of which there is another coefficient, with different indices. In every step, in both of the emerging coefficients, the difference between the first and the second index is decreased by one. However, in the second term, the first index is decreased each time by two and the second by one, while in the first term only the first index is decreased by one. In order for $b_{2\rho - 1, \rho + 1} = 0$ to be zero, all of the emerging coefficients in this recursion must be also zero. Equivalently, the indices of all the emerging coefficients must satisfy one of the conditions mentioned above.

We can represent this recursion as a binary tree, see Figure D.1, each node of which will represent a different coefficient. For the root of this tree to be 0, all the leafs have to be zero too. However, if we prove that the coefficient emerging always from the second term of the recursive equation (4.13), that is the rightmost leaf of the tree is equal to 0 then, all the other coefficients will be equal to 0 as well, because even though the difference in the indexes is decreased by the same amount in each recursion (each level of the tree), the indices of the rightmost coefficients (the nodes in the path from the root to the rightmost coefficient) approach the limit $i = 2$ faster than the nodes in any other path.

After $x$ steps of the recursion, the indices of the rightmost leaf will be

$$b_{i_x, n_x} = b_{2\rho - 1 - 2x, \rho + 1 - x}$$
Figure D.1: Graphical representation of the iterative generation of $b_{i,n}$

The indices have to be equal, in order for the coefficient to be zero, so

$$2\rho - 1 - 2x = \rho + 1 - x \Rightarrow x = \rho - 2$$

Therefore, after $x = \rho - 2$ steps of the recursion, indices will be

$$i_x = 2\rho - 1 - 2(\rho - 2) = 3 > 2$$

$$n_x = \rho + 1 - (\rho - 2) = 3 > 2$$

Thus, it has been proven, that all the coefficients for this term are 0 and the proof for the coordinate vectors is complete. The proof for the curvatures is straightforward using Equation (4.12), since this formula is based on the formula of the coordinate vectors, but for the shake of brevity, only a short proof for the independence of the arc length will be presented next, which is what is required for the curve $A$ to be a hyperhelix.

By definition

$$\kappa_i^2(s) = \|u_i'(s) + \kappa_{i-1}(s)u_{i-1}(s)\|^2 =$$

$$u_i'(s)^H u_i'(s) + \kappa_{i-1}^2(s)u_{i-1}(s)^H u_{i-1}(s) + 2\kappa_{i-1}(s)\text{Re}\{u_{i-1}(s)^H u_i'(s)\}$$

(D.3)

However from Equation (D.1) it is known that the $i$-th coordinate vector $u_i(s)$
can be written as

$$\mathbf{w}_i(s) = \mathbf{x}_i \odot \mathbf{a}(s)$$

where

$$\mathbf{x}_i \triangleq \frac{j^i}{\kappa_1 \kappa_2 \ldots \kappa_{i-1}} \left( \left\lfloor \frac{i-1}{2} \right\rfloor + 1 \sum_{n=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor + 1} (-1)^{n-1} b_{i-1,n} e^{i-2n+2} \right)$$

is independent of $s$. Based on this and on the fact that by assumption $\mathbf{a}'(s) = -j \mathbf{a}(s)$, then all the terms of Equation (D.3) are independent of $s$ provided that $\kappa_{i-1}(s)$ is. However, $\kappa_1, \kappa_2$ are independent of $s$, as can be seen in Equation (4.10) and by induction all the consequent are as well.
Appendix E

Approximate Projection Matrix

Throughout this Appendix the following additional notational conventions will be used.

\[ S_i \triangleq S(r, \theta_i) \]
\[ S_{oi} \triangleq S(r_o, \theta_i) \]
\[ \dot{S}_{oi} \triangleq \frac{\partial S_{oi}}{\partial r} \in \mathbb{C}^{N \times 2N} \]
\[ \dot{S}_{oi}[k, l] = \frac{\partial S_{oi}[k]}{\partial r[l]} \]

where \( \dot{S}_{oi}[k, l] \), \( S_{oi}[k] \) and \( r[l] \) are the \((k, l)\), \(k\)-th and \(l\)-th elements of \( \dot{S}_{oi} \), \( S_{oi} \) and \( r \) respectively.

In the case where only two sources are present, the projection matrix on the signal subspace \( \mathbb{P}_s \) can be written as

\[ \mathbb{P}_s = [S_1 S_2] \left( [S_1 S_2]^H [S_1 S_2] \right)^{-1} [S_1 S_2]^H \quad (E.1) \]
However

\[
\left( [S_1 S_2]^H [S_1 S_2] \right)^{-1} = \frac{1}{\det \left\{ \begin{bmatrix} S_1^H S_1 & S_1^H S_2 \\ S_2^H S_1 & S_2^H S_2 \end{bmatrix} \right\}} \begin{bmatrix} S_2^H S_2 & -S_1^H S_2 \\ -S_2^H S_1 & S_1^H S_1 \end{bmatrix}
\]

(E.2)

Therefore, \( P_s \) can be expressed as

\[
P_s = \frac{1}{N^2 - |S_1^H S_2|^2} \left[ S_1 \begin{bmatrix} S_1^H S_2 & -S_1^H S_2 \\ -S_2^H S_1 & S_1^H S_1 \end{bmatrix} \begin{bmatrix} S_1^H \\ S_2^H \end{bmatrix} \right]
\]

\[
= \frac{N S_1 - (S_2^H S_1) S_2 N S_2 - (S_1^H S_2) S_1}{N^2 - |S_1^H S_2|^2}
\]

(E.3)

Using the following first-order Taylor expansions around the nominal array geometry \( r_o \),

\[
S_1 \approx S_o + \dot{S}_1 \Delta r
\]

(E.4a)

\[
S_2 \approx S_o + \dot{S}_2 \Delta r
\]

(E.4b)

and ignoring any second order terms involving the uncertainties in the array geometry the following approximations can be derived.

\[
(S_1 S_1^H) \approx S_o S_o^H + \dot{S}_o \Delta r S_o^H + S_o \Delta r^T \dot{S}_o^H
\]

(E.5a)

\[
(S_2^H S_1) S_2 \approx \left( S_o^H S_o + S_o^H \dot{S}_o \Delta r + \Delta r^T S_o^H \dot{S}_o \right)
\]

(E.5b)
\[(S^H_1 S_2) S^H_2 \geq \left( S^H_{o1} S_{o2} + S^H_{o1} \dot{S}_{o2} \Delta r + \Delta r^T S^H_{o1} S_{o2} \right)
+ \left( S^H_{o1} S_{o2} + S^H_{o1} \Delta r^T \dot{S}^H_{o2} + \dot{S}_{o1} \Delta r S^H_{o2} \right) \tag{E.5c} \]
\[
S^H_2 S_2 \geq S^H_{o2} S_{o2} + \dot{S}^H_{o2} \Delta r S^H_{o2} + S^H_{o2} \Delta r^T \dot{S}^H_{o2} \tag{E.5d} \]
\[
|S^H_1 S_2|^2 \leq \left( S^H_{o1} S_{o2} + S^H_{o1} \dot{S}_{o2} \Delta r + \Delta r^T \dot{S}^H_{o1} S_{o2} \right)^H.
+ \left( S^H_{o1} S_{o2} + S^H_{o1} \dot{S}_{o2} \Delta r + \Delta r^T \dot{S}^H_{o1} S_{o2} \right) \leq |S^H_{o1} S_{o2}|^2 + S^H_{o2} S_{o1} \Delta r^T \dot{S}^H_{o1} S_{o2} + S^H_{o2} S_{o1} \dot{S}^H_{o1} S_{o2} \Delta r^T \dot{S}^H_{o1} S_{o2} \tag{E.5e} \]
\[
+ S^H_{o2} S_{o1} \dot{S}^H_{o1} \Delta r + S^H_{o2} \Delta r^T \dot{S}^H_{o1} S_{o1} \]
\[
= |S^H_{o1} S_{o2}|^2 + 2 \text{Re} \left\{ S^H_{o2} S_{o1} \Delta r^T \dot{S}^H_{o1} S_{o2} \right\} +
+ 2 \text{Re} \left\{ S^H_{o2} S_{o1} \dot{S}^H_{o1} \Delta r \right\} \]

Substitution of Equations (E.5) into Equation (E.3) will yield the required expression.
Bibliography


Bibliography


[34] A. Manikas, H. R. Karimi, and I. Dacos, “Study of the detection and resolution capabilities of a one-dimensional array of sensors by using differential


[59] B. Wang, Y. Wang, and H. Chen, “Array calibration of angularly dependent gain and phase uncertainties with instrumental sensors,” in *IEEE Interna-


