Existence and uniqueness of hyperhelical array manifold curves

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***Abstract*—A number of significant problems, arising frequently in array signal processing, have been successfully tackled using methods based on the concept of the array manifold. These approaches take advantage of the inherent information about the array system which is encapsulated in the geometry of the array manifold. Array ambiguities, array uncertainties, array design and performance characterisation are just some of the areas that have benefited from this approach. However, the investigation of the geometry of the array manifold itself for most array geometries has been proven to be a complex problem, especially when higher order geometric properties need to be calculated. Nevertheless, special array geometries have been identified, for which the array manifold curve assumes a specific "hyperhelical" shape. This property of the array manifold greatly simplifies its geometric analysis and, consequently, the analysis of the associated array geometries. Hence, the goal of this paper is twofold; to provide the necessary and sufficient conditions for the existence of array manifold curves of hyperhelical shape; and to determine which array geometries can actually give rise to manifold curves of this shape.**

***Index Terms*—array manifolds, array processing, differential geometry, hyperhelices, array design**

NOTATION

a, A	scalar
$\underline{a}, \underline{A}$	column vector
$\mathbb{A}$	matrix
$\underline{0}_N$	N element column vector of zeros
$\underline{1}_N$	N element column vector of ones
$\otimes$	Kronecker product
$\odot$	Hadamard product
$\lfloor \cdot \rfloor$	round down to integer
$\Vert\underline{a}\Vert$	Euclidean norm of the column vector $\underline{a}$
sum $\{\underline{a}\}$	sum of the entries of the column vector $\underline{a}$
$\underline{a}^n$	element-wise

I. INTRODUCTION

A wide variety of array signal processing algorithms employ the concept of the array manifold vector (also known as array response vector), to perform classical array processing tasks, including but not limited to source detection and channel estimation (e.g. the well known MUSIC algorithm [1] and variants of it [2]), signal reception [3], array calibration [4], source localisation [5], antenna-array design [6], array ambiguities mitigation [7] etc. Based on the success of the the models employing the array manifold vector, the concept of the array manifold was introduced. The array manifold is the locus of all possible array manifold vectors as the channel parameters (e.g. azimuth, delay, carrier frequency, Doppler spread, signal polarisation etc) vary. The intrinsic geometric properties of the *array manifold* encapsulate all of the information about the properties of the array system, as well as its operational characteristics.

The most commonly used array manifold is the spatial array manifold, which for an array of N omnidirectional elements is defined as

$$\mathcal{M} \triangleq \left\{ \underline{a}(p,q) \in \mathcal{C}^N \ , \forall (p,q) \ : \ (p,q) \in \Omega \right\} \tag{1}$$

where p, q are directional parameters of the incoming sources (e.g. azimuth and elevation angles of arrival, or directional cosines), Ω is the parameter space, and

$$\underline{a}(p,q) = \exp\left(-j \ \left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right] \ \underline{k}(p,q)\right) \tag{2}$$

is the spatial array manifold vector, where $\left[\underline{r}_x \ , \ \underline{r}_y \ , \ \underline{r}_z\right] \in \mathcal{R}^{N \times 3}$ is the array containing the x, y, z coordinates of the N antenna array elements in units of half wavelengths and

$$\underline{k}(p,q) = \pi \left[\cos(p)\cos(q) \ , \ \sin(p)\cos(q) \ , \ \sin(q)\right]^T$$

is the vector pointing in the direction of the (p,q) source.

Any information included in the array manifold vector is also incorporated in the geometric properties of the array manifold. The differential geometry approach, which studies the array manifold as a geometric object embedded in a proper multidimensional complex space has led to a number of significant results regarding the performance analysis of array systems. The problem of ambiguities in array processing has been analysed and linked with array manifold geometric properties in [8]. In [9], theoretical lower bounds regarding the detection, resolution and accuracy capabilities of linear and some planar array systems have been derived as function of the intrinsic geometry of the spatial array manifold. Moreover, in [10] these results were extended to the case of any 3-dimensional array geometry and then to array systems modelled using the extended array manifolds enabling for a comprehensive study of practical array systems. In [11] geometric concepts have been utilised to tackle the array design problem in an optimal way. Geometric information about the array manifold was also used in [12], where the author utilised the array manifold geometry to perform array design under accuracy and peak side-lobe level constraints and in [6] for array manifold interpolation. In addition, in [13] the authors study the array geometries which give rise to isotropic array performance with regards to the signal DOA.

These promising results have motivated the introduction of more general array manifold models, such as the "extended array manifold" defined in [10] and a generalised array manifold for the modelling of local scatterers in [14]. In addition, is a series of papers [15], [16], [17], the authors therein proposed a new approach to representing and simplifying the analysis of the array manifold.

Although the differential geometric approach has been shown to bear fruits in a number of areas, the geometric analysis of the array manifolds for most array geometries can be extremely complex due to the high degree of non-linearity of the associated array manifolds. This, in turn, renders the derivation of the aforementioned theoretical results a challenging problem by itself and usually one is compelled to resort to lower-rank approximations. However, in [18] a specific class of spatial array manifold curves (array manifolds depending on a single parameter of interest) have been identified, the so-called "hyperhelical array manifold curves", the analysis of which is greatly simplified. In detail, array manifold curves belonging in this class have constant curvatures throughout the length of the curve, which greatly facilitates their geometric analysis. For hyperhelical manifold curves, analytic expressions for the most significant theoretical results and bounds can be and have been derived [18].

In addition, in [10] the concept of hyperhelix was carried over to additional array manifolds, besides the spatial one defined in Eq. (2), through the concept of the extended array manifold. Although, we will not make use of the concept of the extended array manifold in this work, for reasons of completeness, we would like to mention that the extended array manifold is defined as a linear complex mapping of the spatial one. It was shown in [10] that if the spatial array manifold curve has constant curvatures, then, based on some quite general assumptions about the nature of this complex mapping, the extended array manifold curve has constant curvatures as well. Therefore, in the cases of extended array manifolds for which these assumption hold, the analysis of the extended array manifold curves is simplified as well. These properties render the hyperhelical manifold curves extremely useful and given their importance, it is interesting to determine which array geometries can, in theory, give rise to such curves.

Hence, the purpose of this paper is to investigate the necessary and sufficient conditions for the existence of hyperhelical spatial array manifold curves in an array of a given geometry. After these conditions have been identified, the next goal is to determine all the possible array geometries or classes of array geometries, which, through the parameterisation of Eq. (2), can give rise to hyperhelices.

This paper is organised as follows. In Section II, some basic notions and definitions of the differential geometry of space curves are given in order to render this research work as self-contained as possible. Then, in Section III the most general *natural* representation of space curves in complex spaces is derived. In Section IV we examine how this *natural* representation and physical assumptions about the array system restrict the possible *regular* representations of hyperhelical spatial array manifold curves. Next, in Section V we determine which array geometries can give rise to such *regular* parametric representations. Finally, the paper is concluded in Section VI.

II. Hyperhelical Space Curves Embedded in $\mathcal{C}^N$

Prior to presenting the main theoretical results of this research work, a short definition of some notions of differential geometry, which will be used frequently in what follows, will be given.

A *regular parametric representation* of a curve $\mathcal{A}$ embedded in an N-dimensional complex space ($\mathcal{C}^N$) is a complex valued vector function

$$\underline{a} = \underline{a}\,(p) \in \mathcal{C}^{N\times 1} \ , \ p \in \Omega \subseteq \mathcal{R} \tag{3}$$

of the parameter p, defined in an interval $\Omega \subseteq \mathcal{R}$, with the following properties[1].

$$\begin{aligned} &\underline{a}\,(p) \ \text{ is of class } C^1 \text{ in } \Omega \\ &\dot{\underline{a}}(p) \triangleq \frac{d\underline{a}\,(p)}{dp} \neq \underline{0} \ , \ \forall p \in \Omega \end{aligned} \tag{4}$$

A real valued function $\mathrm{p} = \mathrm{p}(p) : \Omega \rightarrow \Omega$ is an *allowable change of parameter* for the expression of $\mathcal{A}$ if

$$\begin{aligned} &\mathrm{p}(p) \text{ is of class } C^1 \text{ in } \Omega \\ &\frac{d\mathrm{p}}{dp} \neq 0 \ , \ \forall p \in \Omega \end{aligned} \tag{5}$$

Two regular representations $\underline{a}(\mathrm{p})$ and $\underline{a}(p)$ are *equivalent* if and only if there exists an allowable change of parameter $\mathrm{p} = \mathrm{p}(p)$ such that

$$\underline{a}\,(\mathrm{p}\,(p)) = \underline{a}(p), \ \forall p \in \Omega$$

What has to be pointed out is that a curve $\mathcal{A}$ may have many equivalent regular parametric representations, but **the properties of the curve are independent of the parameter**.

Assume that $\underline{a} = \underline{a}(p)$ is a regular parametric representation of the curve $\mathcal{A}$ and let $s = s(p)$ define the arc length of the curve, as it is measured from a given point $\underline{a}\,(p_o)$ on the curve, that is

$$s = s\,(t) = \int_{p_o}^{p} \|\dot{\underline{a}}\,(\xi)\|\, d\xi \tag{6}$$

However from Eq. (4), $\dot{s}(p) \equiv \|\dot{\underline{a}}\,(p)\|$ is continuous and non-zero $\forall p \in \Omega$. Thus, $s = s(p)$ is an allowable change of parameter on Ω and $\underline{a} = \underline{a}(s)$ is a regular parametric representation of $\mathcal{A}$. Furthermore,

$$\|\underline{a}'(s)\| \triangleq \left\| \frac{d\underline{a}\,(s)}{ds} \right\| = 1$$

so that $\underline{a} = \underline{a}(s)$ is called a *natural representation* of $\mathcal{A}$.

A very important concept of Differential Geometry related to curves is that of their curvatures. According to the Fundamental Uniqueness Theorem by Gauss, a space curve expressed in a *natural representation*, (i.e. in term of its arc length) is uniquely defined by its curvatures, except for its position in space. For a curve $\mathcal{A}$ of the form of Eq. (3) defined in $\Omega = [p_{min}, p_{max}]$ and an allowable change of parameter of the form of Eq. (6), an *equivalent*, natural representation is the following

$$\underline{a} = \underline{a}(s) \in \mathcal{C}^{N\times 1} \ , \quad s(p_{min}) = 0 \leq s \leq l_m \triangleq s(p_{max}) \tag{7}$$

[1]A function is said to be of class C^1 if its first derivative exists and is continuous

A curve embedded in an N-dimensional complex space has $2N-1$ non-zero curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{2N-1}$. For the sake of completeness, note that these are actually defined in the isomorphic $2N$-dimensional real space, although this is not of importance in this work and, henceforth, we will not make any distinction between the two spaces. However, if this curve is limited to a subspace $\mathcal{C}^u$ of $\mathcal{C}^N$, then only $d-1 = 2u-1$ non-zero curvatures can be defined. A (real) example of this is a curve lying on the plane $x = y$ of $\mathcal{R}^3$. Although the dimensionality of the real space is 3, the curve is limited on a plane, the dimensionality of which is obviously equal to 2 and, hence, only κ_1 is non-zero.

The curvatures of a space curve $\mathcal{A}$ embedded in a subspace of $\mathcal{C}^N$ are defined using the concept of the Frenet Frame, which is a set of d complex unit vectors $\underline{u}_i(s)$, $i = 1, \ldots, d$, which are attached at each point $\underline{a}(s)$ of the curve and serve as a local coordinate system. The Frenet vectors and the curvatures are given by [19]

$$\left.\begin{aligned}
\underline{u}_1(s) &= \underline{a}'(s)\\
\kappa_1(s) &= \|\underline{u}_1'(s)\|\\
\underline{u}_2(s) &= \frac{\underline{u}_1'(s)}{\kappa_1(s)}\\
\kappa_2(s) &= \|\underline{u}_2'(s) + \kappa_1(s)\underline{u}_1(s)\|\\
&\cdots \quad \cdots\\
\underline{u}_i(s) &= \frac{\underline{u}_{i-1}'(s) + \kappa_{i-2}(s)\underline{u}_{i-2}(s)}{\kappa_{i-1}(s)}\\
\kappa_i(s) &= \left\|\underline{u}_i'(s) + \kappa_{i-1}(s)\underline{u}_{i-1}(s)\right\|\\
&\cdots \quad \cdots
\end{aligned}\right\} \tag{8}$$

The curvatures are, in the general case functions of the arc length s. We define as *hyperhelix* the space curve $\mathcal{A}$ which has constant curvatures, that is

$$\frac{d\kappa_i(s)}{ds} = 0 \Rightarrow \kappa_i(s) = \kappa_i \;,\; i = 1, \ldots, d \tag{9}$$

III. Natural Representation of Hyperhelices

So far, in [18] it has been shown that the spatial array manifold curves of linear arrays and the elevation spatial manifold curves of planar arrays are hyperhelices. The regular parametric representations of these two classes of curves present some distinct similarities. Based on this observation, the question naturally arises of whether there is a more general equation for the representation of a hyperhelical manifold curve embedded in $\mathcal{C}^N$ and whether this equation is unique, i.e.

- Is there an equation which may serve as a general regular parametric representation of a hyperhelical manifold curve (not necessarily an array manifold curve but any space curve) and which encompasses the known formula of hyperhelical array manifold curve as a special case?
- If such a representation does exist, is it unique?

Theorem 1, which follows, is the first step towards answering the previous questions. It provides a general formula for a natural representation of a hyperhelical manifold curve in $\mathcal{C}^N$ and asserts that this representation in terms of the arc length s of the curve is unique.

Theorem 1. *Let a space curve $\mathcal{A}$ embedded in a u-dimensional complex space $\mathcal{C}^u$ and let $\mathcal{C}^u$ be the space of minimum dimensionality which contains $\mathcal{A}$. Then, $\mathcal{A}$ is a hyperhelix, that is the curvatures of $\mathcal{A}$ are constant, if and only if there is a constant real vector $\underline{\tilde{r}} \in \mathcal{R}^{d\times 1}$ and a constant complex vector $\underline{v} \in \mathcal{C}^{d\times 1}$, with*

$$d = 2u \tag{10a}$$

$$\|\underline{v} \odot \underline{\tilde{r}}\| = 1 \tag{10b}$$

$$\mathit{sum}\{\underline{\tilde{r}}\} = 0 \tag{10c}$$

such that

$$\underline{a}(s) = \underline{v} \odot \exp\{-j\underline{\tilde{r}} \cdot s\} \tag{11}$$

is a natural representation of $\mathcal{A}$.

Proof:

Throughout this proof, x' will denote differentiation of the quantity x with respect to the arc length s.

Note that Eq. (10b) guarantees that Eq. (11) is a natural representation, since

$$\|\underline{a}'(s)\| = \|\underline{v} \odot \underline{\tilde{r}}\| = 1$$

Forward: Let us assume that Eq. (11) is a natural representation of the curve $\mathcal{A}$. Then, it is sufficient to show that its curvatures are independent of the arc length s. Based on Eq. (8), the lower order coordinate vectors and curvatures are given below.

$$\left.\begin{aligned}
&\underline{u}_1(s) = \underline{a}'(s) = -j\;\underline{\tilde{r}} \odot \underline{a}(s)\\
&\underline{u}_1'(s) = -\underline{\tilde{r}}^2 \odot \underline{a}(s) \qquad\qquad \kappa_1 = \left\|\underline{v} \odot \underline{\tilde{r}}^2\right\|\\
&\underline{u}_2(s) = -\frac{1}{\kappa_1}\underline{\tilde{r}}^2 \odot \underline{a}(s)\\
&\underline{u}_2'(s) = j\underline{\tilde{r}}^3 \odot \underline{a}(s) \quad \kappa_2 = \frac{1}{\kappa_1}\left\|\underline{v} \odot \left(\underline{\tilde{r}}^3 - \kappa_1^2\underline{\tilde{r}}\right)\right\|\\
&\cdots \quad \cdots
\end{aligned}\right\} \tag{12}$$

Based on the results for the first and second coordinate vectors, the emerging pattern of the formulae leads to the the following expressions

$$\begin{aligned}
\underline{u}_i(s) = \frac{\underline{u}_{i-1}' + \kappa_{i-2}\underline{u}_{i-2}}{\kappa_{i-1}} = \frac{j^i}{\kappa_1 \ldots \kappa_{i-1}}&\\
\left[\left(\sum_{n=1}^{\lfloor\frac{i-1}{2}\rfloor+1} (-1)^{n-1}\, b_{i-1,n}\;\underline{\tilde{r}}^{i-2n+2}\right) \odot \underline{a}(s)\right]&
\end{aligned} \tag{13}$$

$$\begin{aligned}
\kappa_i = \|\underline{u}_i' + \kappa_{i-1}\cdot\underline{u}_{i-1}\| = \frac{1}{\kappa_1\kappa_2\ldots\kappa_{i-1}}\cdot&\\
\cdot\left\|\underline{v} \odot \sum_{n=1}^{\lfloor\frac{i}{2}\rfloor+1} (-1)^{n-1}\, b_{i,n}\;\underline{\tilde{r}}^{i-2n+3}\right\|&
\end{aligned} \tag{14}$$

where

$$b_{i,n} = \quad b_{i-1,n} + \kappa_{i-1}^2 \cdot b_{i-2,n-1} \quad , \; i \geq 1 \tag{15}$$

with

$$\begin{aligned}
b_{i,1} &= 1 \;,\; i \geq 1\\
b_{2,2} &= \kappa_1^2
\end{aligned} \tag{16}$$

The proof of Eqs. (13) - (16) can be found in Appendix A. Since the curve is embedded in a u-dimensional complex space, these formulae are valid for $i = 1, \ldots, d-1 = 2u-1$ since the d-th curvature is zero for any curve, not only a hyperhelix, constrained in a u-dimensional complex space. The important characteristic of Eq. (14) is that it is independent of the parameter s. Hence, by the definition of hyperhelix, curve $\mathcal{A}$ is of hyperhelical shape.

Converse: Let as assume now that $\mathcal{A}$ is a hyperhelix, embedded in the u-dimensional complex space $\mathcal{U}$. Since by assumption $\mathcal{U}$ is the space of minimum dimensionality that contains $\mathcal{A}$, the latter has $d = 2u$ constant curvatures $\kappa_1, \kappa_2, \ldots, \kappa_d$, of which the first $d-1$ are non-zero and of course $\kappa_d = 0$. Let us consider the set $\mathcal{S}_d^\infty$ of all the possible hyperhelices embedded in the same complex space $\mathcal{U}$. Each hyperhelix $\mathcal{A}_{\underline{\kappa}}^{\text{helix}}$ in $\mathcal{S}_d^\infty$ is uniquely defined, according to the Fundamental Uniqueness theorem by Gauss, by a vector $\underline{\kappa} = [\kappa_1, \kappa_2, \ldots, \kappa_{d-1}, 0]^T$ of curvatures.

Consider now the set $\mathcal{S}_d^{\tilde{r}}$ of those hyperhelices which can be expressed by a natural representation in the form of Eq. (11). It was shown previously that $\mathcal{S}_d^{\tilde{r}} \subseteq \mathcal{S}_d^\infty$. The proof will be complete if it is shown that $\mathcal{S}_d^{\tilde{r}} = \mathcal{S}_d^\infty$.

To that end, let us choose one of the members $\mathcal{A}_{\underline{\kappa}}^{\text{helix}}$ of $\mathcal{S}_d^\infty$, which is defined by a vector $\underline{\kappa} = [\kappa_1, \kappa_2, \ldots, \kappa_{d-1}, 0]^T$ of constant curvatures. The objective is to find a constant real vector $\underline{\tilde{r}} \in \mathcal{R}^{N\times 1}$ and a complex constant vector $\underline{v} \in \mathcal{C}^{N\times 1}$, such that the curve $\mathcal{A}$ having as a natural representation the vector function $\underline{\mathrm{a}}(s) = \underline{v} \odot \exp\{-j\underline{\tilde{r}} \cdot s\}$ has the same curvatures as $\mathcal{A}_{\underline{\kappa}}^{\text{helix}}$.

Let us write down the Frenet-Serret formulae [19] which connect the coordinate vectors $\underline{u}_i,\ i = 1, \ldots, d$ and curvatures $\kappa_i,\ i = 1, \ldots, d$ of a space curve.

$$\left(\mathbb{U}(s)^T\right)' = \mathbb{C}(s)\,\mathbb{U}(s)^T \tag{17}$$

where

$$\mathbb{C}(s) = \begin{bmatrix} 0 & \kappa_1(s) & \ldots & 0 & 0 \\ -\kappa_1(s) & 0 & \ldots & 0 & 0 \\ 0 & -\kappa_2(s) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \kappa_{d-1}(s) \\ 0 & 0 & \ldots & -\kappa_{d-1}(s) & 0 \end{bmatrix} \tag{18}$$

is the Cartan matrix and

$$\mathbb{U}(s) = [\underline{u}_1(s), \ldots, \underline{u}_d(s)] \tag{19}$$

Let us denote as

$$\mathbb{Y}(s) \triangleq \mathbb{U}^T(s) = \left[\underline{y}_1(s), \ldots, \underline{y}_N(s)\right] \tag{20}$$

where

$$\underline{y}_i(s) \triangleq [U_{i1}(s), U_{i2}(s), \ldots, U_{id}(s)]^T$$

the i-th column of $\mathbb{U}^T$. Then, Eq. (17) can be re-written as

$$\mathbb{Y}'(s) = \mathbb{C}(s)\,\mathbb{Y}(s) \tag{21}$$

Eq. 21 corresponds to d decoupled systems of first order differential equations. In this case the Cartan matrix is independent of s, so that all $\underline{y}_i(s),\ i = 1, \ldots, N$ satisfy the following differential equation with constant coefficients

$$\underline{y}_i'(s) = \mathbb{C}\ \underline{y}_i(s) \tag{22}$$

Since $\mathbb{C}$ is a skew-Hermitian matrix, it is normal and therefore has a set of d eigenvalues $\lambda_i, i = 1, \ldots, d$ which are purely imaginary and come in conjugate pairs, and d independent eigenvectors $\underline{E}_i, i = 1, \ldots, d$ forming the matrix

$$\mathbb{E} \triangleq [\underline{E}_1, \ldots, \underline{E}_d] \tag{23}$$

Hence, Eq. (22) has d independent solutions given by

$$\underline{y}_i(s) = \underline{E}_i \exp\{\lambda_i s\},\ i = 1, \ldots, d \tag{24}$$

Hence, the general solution of Eq. (21) is

$$\mathbb{Y} = \mathbb{E}\ \text{diag}\{\exp\{\underline{\lambda} s\}\} \tag{25}$$

where $\underline{\lambda}$ is the vector of the eigenvalues of the Cartan matrix.

Consequently, the matrix of the coordinate vectors $\mathbb{U}$ is given by

$$\mathbb{U} = \text{diag}\{\exp\{\underline{\lambda} s\}\}\ \mathbb{E}^T \tag{26}$$

and the first coordinate vector $\underline{u}_1(s)$ is given by

$$\underline{u}_1(s) = \underline{\mathrm{E}}_1 \exp\{\underline{\lambda} s\} \tag{27}$$

where $\underline{\mathrm{E}}_1$ is the first row of $\mathbb{E}$. Therefore, the natural representation of the curve can be found by integrating Eq. (27) which yields

$$\underline{\mathrm{a}}(s) = (\underline{\mathrm{E}}_1 \oslash \underline{\lambda}) \odot \exp\{\underline{\lambda} s\} \tag{28}$$

From Eq. (28), we conclude that

$$\underline{v} = \underline{\mathrm{E}}_1 \oslash \underline{\lambda} \tag{29}$$

$$\underline{\tilde{r}} = j\underline{\lambda} \tag{30}$$

Note that

$$\|j(\underline{\mathrm{E}}_1 \oslash \underline{\lambda}) \odot \underline{\lambda}\| = 1 \tag{31}$$

and

$$\text{sum}\{\underline{\lambda}\} = 0 \tag{32}$$

since the eigenvalues of a skew-symmetric matrix like $\mathbb{C}$ are purely imaginary and come in conjugate pairs. In addition, none of the eigenvalues can be zero, because that would imply that the Cartan matrix would have a determinant equal to zero. However

$$\det(\mathbb{C}) = \kappa_1 \kappa_2 \ldots \kappa_{d-1}$$

Thus, this would signify that some of the first $d-1$ curvatures is zero, which is contrary to the assumptions made so far. ■

A. Comments on Theorem 1

Theorem 1 states that given d constant curvatures it is always possible to have a complex vector $\underline{a}(s) \in \mathcal{C}^d$ of the form of Eq. (11), which will be the natural representation of a curve represented by these curvatures. In practice, however, for array manifold curves it has been shown [20] that it is possible to have an array with N elements, the array manifold curve of which will be embedded in a u-dimensional complex space, where $N < 2u = d$. It is useful from an array design point of view to see how this may arise in the framework of the analysis presented in this section.

As was stated before, the Cartan matrix $\mathbb{C}$ is a skew-symmetric matrix and therefore its curvatures are purely imaginary and come in conjugate pairs, which implies that

$$\text{sum}\{\underline{\lambda}\} = 0 \tag{33}$$

Let us consider the set of $d/2$ curvatures with distinct absolute value $|\lambda_i|$. If there is a subset of this set, with m eigenvalues $\lambda_1, \ldots, \lambda_m$ such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m = 0 \tag{34}$$

then this implies that the m corresponding entries $\exp\{\lambda_1 s\}, \ldots, \exp\{\lambda_m\} s$ are linearly dependent and are, therefore, redundant in the description of the manifold curve. Hence the dimension of the vector $\underline{a}(s)$ can be reduced from d to $d - m$ by removing these redundant entries.

Next two examples are going to be given, which will clarify these concepts.

1) Example 1: Let us consider the following set of curvatures.

$$\underline{\kappa} = [0.5, 0.4, 0.3, 0.2, 0.1, 0]^T$$

The eigenvalues of the Cartan matrix are

$$\underline{\lambda} = j\,[\pm 0.6742, \pm 0.3, \pm 0.0742]^T$$

No subset of these eigenvalues adds up to 0, so the natural representation of this curve is (see Eq. (28))

$$\underline{a}(s) = \begin{bmatrix} 0.4917j \\ -0.4917j \\ 0.4029j \\ -0.4029j \\ -0.3097 \\ 0.3097j \end{bmatrix} \oslash \underline{\lambda} \odot e^{\underline{\lambda} s}$$

2) Example 2: Let us consider the following set of curvatures.

$$\underline{\kappa} = [0.6019, 0.2076, 0.2974, 0.1185, 0.2089, 0.1264, 0.1224, 0]^T$$

The eigenvalues of the Cartan matrix are

$$\underline{\lambda} = j\,[\pm 0.6462, \pm 0.3231, \pm 0.2261, \pm 0.0969]^T$$

Obviously, $0.3231 - 0.2261 - 0.0969 = 0$. So $m = 3$ of the eigenvalues add up to 0 and therefore the following two representations are equivalent.

$$\underline{a}_1(s) = \begin{bmatrix} -j \\ -j \\ -0.7073j \\ -0.7073j \\ -0.7068j \\ -0.7068j \\ -0.7071j \\ -0.7071j \end{bmatrix} \oslash \underline{\lambda} \odot \exp\{\underline{\lambda} s\}$$

and

$$\underline{a}_2(s) = \begin{bmatrix} -j \\ -j \\ -0.7073j \\ -0.7068j \\ -0.7071j \end{bmatrix} \oslash \begin{bmatrix} j0.6462 \\ -j0.6462 \\ -j0.3231 \\ j0.2261 \\ j0.0969j \end{bmatrix} \odot \begin{bmatrix} \exp\{j0.6462s\} \\ \exp\{-j0.6462s\} \\ \exp\{-j0.3231s\} \\ \exp\{j0.2261s\} \\ \exp\{j0.0969s\} \end{bmatrix}$$

IV. Regular Parametric Representation of Hyperhelices in $\mathcal{C}^N$

In the previous section, Theorem 1 provided the most general form of a natural representation of a hyperhelical manifold curve in $\mathcal{C}^N$. However, the ultimate goal of this work is to identify which array geometries can, potentially, give rise to hyperhelical spatial array manifold curves. Therefore, we proceed now into investigating what Theorem 1, which is valid for any manifold curve in $\mathcal{C}^N$, implies in the case of array manifold curves.

Using the azimuth θ and elevation ϕ as parameters, the spatial array manifold vector of an array of N omnidirectional elements (defined in Eq. (2)) can be written as

$$\underline{a}(\theta,\phi) = \exp\left(-j\left[\underline{r}_x\ ,\ \underline{r}_y\ ,\ \underline{r}_z\right]\ \underline{k}(\theta,\phi)\right) \tag{35}$$

where $\left[\underline{r}_x\ ,\ \underline{r}_y\ ,\ \underline{r}_z\right]$ has been defined in Section I and

$$\underline{k} = \pi\left[\cos(\theta)\cos(\phi)\ ,\ \sin(\theta)\cos(\phi)\ ,\ \sin(\phi)\right]^T$$

The locus of the array manifold vectors as θ and ϕ vary, forms the array manifold surface

$$\mathcal{M} \triangleq \left\{\underline{a}(\theta,\phi) \in \mathcal{C}^N\ ,\forall(\theta,\phi)\ :\ (\theta,\phi) \in \Omega\right\} \tag{36}$$

where Ω is the parameter space, usually $\theta \in [0, 2\pi)$, $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

By enforcing a constraint

$$F(\theta,\phi) = 0 \tag{37}$$

the array manifold vector traces a curve

$$\mathcal{A} \triangleq \{\underline{a}(\theta,\phi), \forall(\theta,\phi) : F(\theta,\phi) = 0 \wedge (\theta,\phi) \in \Omega\} \tag{38}$$

lying on the array manifold surface of Eq. (36). In this case, the array manifold vector of Eq. (35) can be rewritten as

$$\begin{aligned} \underline{a}(p) &= \exp\left(-j\ \underline{g}(p)\right) \\ \underline{g}(p) &\triangleq \left[\underline{r}_x\ ,\ \underline{r}_y\ ,\ \underline{r}_z\right]\ \underline{k}(p) \end{aligned} \tag{39}$$

provided that $p = p(\theta, \phi) \in \Omega_p$ is an allowable change of parameter. Note that due to the constraint (37), $p = p(\theta, \phi)$ is actually a function of **one** independent variable.

Eq. (39) is a regular parametric representation of the spatial manifold curve $\mathcal{A}$ since $p = p(\theta, \phi) \in \Omega_p$ is an allowable change of parameter. However, p is not necessarily a natural parameter and Theorem 1 cannot be readily applied. For this reason, before investigating issues related to the existence of hyperhelical array manifold curves, it is convenient to investigate what the implications of Theorem 1 are for the possible regular parametric representations of hyperhelical array manifold curves in $\mathcal{C}^N$.

Theorem 2 presented next addresses exactly this issue, as it provides the general form of a regular parametric representation of a hyperhelical array manifold curve.

Theorem 2. *Let $\mathcal{A}$ be the array manifold curve of an array of N-omnidirectional elements. Then, $\mathcal{A}$ is a hyperhelix if and only if there are two constant real vectors $\underline{r} \in \mathcal{R}^{N\times 1}$ and $\underline{K} \in \mathcal{R}^{N\times 1}$ and a real valued scalar function $g(p)$ of the parameter p such, that*

$$\underline{\mathrm{a}}(p) = \exp\left\{-j\left(\underline{r}\ g\left(p\right) + \underline{K}\right)\right\}\ ,\ \in \mathcal{C}^{N\times 1} \tag{40}$$

is a regular parametric representation of $\mathcal{A}$.

PROOF*:*

According to Theorem 1, the array manifold curve $\mathcal{A}$ will be a hyperhelix if and only if there are two constant vectors $\underline{\tilde{r}} \in \mathcal{R}^N$ and $\underline{v} \in \mathcal{C}^N$, such that

$$\underline{\mathrm{a}}(s) = \underline{v} \odot \exp\left\{-j\underline{\tilde{r}} \cdot s\right\} \tag{41}$$

is a *natural representation* of $\mathcal{A}$.

It was shown that any regular parametric representation for the array manifold curve $\mathcal{A}$ can be expressed in the following format

$$\underline{\mathrm{a}}\left(p\right) = \exp\left\{-j\ \underline{g}\left(p\right)\right\} \tag{42}$$

where $\underline{g}\left(p\right) \in \mathcal{R}^N$. This implies that

$$\begin{aligned} &\underline{\mathrm{a}}(p) \text{ is of class } C^1 \text{ in } I \Leftrightarrow \underline{g}(p) \text{ is of class } C^1 \text{ in } I \\ &\underline{\dot{\mathrm{a}}}(p) \triangleq \frac{d\underline{\mathrm{a}}(p)}{dp} \neq \underline{0}\ ,\ \forall p \in I \Leftrightarrow \underline{\dot{g}}(p) \neq \underline{0}\ ,\ \forall p \in I \end{aligned} \tag{43}$$

Since expressions (41) and (42) refer to the same manifold curve, they have to be *equivalent*, which implies that

$$\begin{aligned} &\underline{\mathrm{a}}\left(p\left(s\right)\right) = \underline{\mathrm{a}}\left(s\right) \Rightarrow \\ &\exp\left\{-j\ \underline{g}\left(p\left(s\right)\right)\right\} = \exp\left\{-j\underline{\tilde{r}} \cdot s + \underbrace{\ln\underline{\beta} - j\underline{\omega}}_{\ln\underline{v}}\right\} \end{aligned} \tag{44}$$

where $\ln\underline{v} = \ln\underline{\beta} - j\underline{\omega},\ \beta_i = |v_i|\,,\ -\omega_i = \arg\{v_i\}$, with $\underline{\beta}, \underline{\omega}$ constant real vectors.

Note that the general natural representation of a hyperhelical manifold curve has a complex phase or equivalently an amplitude vector $\underline{\beta}$ different than $\underline{1}_N$. However, since we are interested in spatial array manifold curves, for Eq. (44) to be true, we have to restrict ourselves to the case where

$$\ln\underline{\beta} = \underline{0}_N \tag{45}$$

This restriction reflects the physical assumption that the phase of the array manifold vector needs to be real, or equivalently that the propagation time of the electromagnetic wave between two sensors is real.

Thus, Eq. (44) becomes

$$\exp\left\{-j\ \underline{g}\left(p\left(s\right)\right)\right\} = \exp\left\{-j\left(\underline{\tilde{r}} \cdot s + \underline{\omega}\right)\right\} \tag{46}$$

Eq. (46) constitutes a system of N equations. Let us consider the $i-th$ and the $j-th$ equations of this system

$$\begin{aligned} \tilde{r}_i s + \omega_i &= g_i\left(p\left(s\right)\right) \\ \tilde{r}_j s + \omega_j &= g_j\left(p\left(s\right)\right) \end{aligned} \tag{47}$$

Note that from Theorem 1 the existence of at least one i for which $\tilde{r}_i \neq 0$ is guaranteed. Eq. (47) implies that

$$g_j\left(p\right) = \underbrace{\frac{\tilde{r}_j}{\tilde{r}_i}}_{\triangleq r_j}\ g_i\left(p\right) + \underbrace{\left(\omega_j - \frac{\tilde{r}_j}{\tilde{r}_i}\omega_i\right)}_{\triangleq K_j} \tag{48}$$

Of course, Eq. (48) is true $\forall i, j = 1, \ldots, N$. Thus, $\underline{g}\left(p\right)$ can be written in the form

$$\underline{g}\left(p\right) = \underline{r}\ g\left(p\right) + \underline{K}$$

where $g\left(p\right)$ can be chosen to be any of the different $g_i\left(p\right)\ ,\ i = 1, \ldots, N$, for which $\tilde{r}_i \neq 0$, so that it is not constant in p. Note that the conditions of Eq. (43) guarantee that there is at least one $g_i\left(p\right)$ such that

$$\frac{dg_i\left(p\right)}{dp} \neq 0$$

■

V. POSSIBLE HYPERHELICAL ARRAY MANIFOLD CURVES

Based on the results of the previous sections it is now possible to examine which array geometries may, in principle at least, give rise to spatial array manifold curves, which can be expressed in the form of Eq. (3). Let us write the spatial array manifold vector of an array of N omnidirectional elements as follows.

$$\underline{\mathrm{a}}\left(p\right) = \exp\left\{-j\left[\underline{r}_1, \ldots, \underline{r}_N\right]^T \underline{k}\left(p\right)\right\} \tag{49}$$

where $\underline{r}_i,\ \ i = 1, \ldots, N$ is the coordinates vector of the i-th array element and p is any allowable parameter, so that Eq. (49) is a regular parametric representation of the array manifold curve. The following Theorem provides the necessary conditions for the array geometry, so that Eq. (49) represents a hyperhelical array manifold curve in $\mathcal{C}^N$.

Theorem 3. *Let $\mathcal{A}$ be the spatial manifold curve of an array of N-omnidirectional elements. Then, for $\mathcal{A}$ to be a hyperhelix, it is necessary that the array is either linear or planar.*

Proof:

It has been proven in the previous sections that every hyperhelical manifold curve embedded in a multi-dimensional complex space $\mathcal{C}^N$ can be expressed in the form of the natural representation (11). For hyperhelical manifold curves, this natural representation is *equivalent* to the regular parametric representation (40). Therefore, in order for $\mathcal{A}$ to be of hyperhelical shape, two real vectors $\underline{r}, \underline{K}$ have to exist, such that

$$\begin{aligned}\exp\left\{-j\left[\underline{r}_1,\ldots,\underline{r}_N\right]^T \underline{k}\left(p\right)\right\} = \\ = \exp\left\{-j\left(\underline{r}g\left(p\right)+\underline{K}\right)\right\}, \ \forall p \in \Omega_p\end{aligned} \quad (50)$$

By considering two arbitrary rows i, j of the vector equation (50)

$$\left.\begin{aligned}\underline{r}_i^T \ \underline{k}\left(p\right) = r_i g\left(p\right) + K_i \\ \underline{r}_j^T \ \underline{k}\left(p\right) = r_j g\left(p\right) + K_j\end{aligned}\right\} \quad (51)$$

which implies that

$$\begin{aligned}&\frac{\underline{r}_i^T \ \underline{k}\left(p\right) - K_i}{\underline{r}_j^T \ \underline{k}\left(p\right) - K_j} = \frac{r_i}{r_j} \Rightarrow \\ &\underline{r}_j^T \ \underline{k}\left(p\right) = C_1 \ \underline{r}_i^T \underline{k}\left(p\right) + C_2, \ \forall p \in \Omega_p\end{aligned} \quad (52)$$

However $\underline{r}_i^T \ \underline{u}\left(p\right)$ is the norm of the projection of $\underline{r}_i$ onto $\underline{k}\left(p\right)$ and similarly for $\underline{r}_j^T \ \underline{k}\left(p\right)$. Thus, the only cases in which Eq. (52) holds $\forall p \in \Omega_p$, as $\underline{k}(p)$ moves in the 3-dimensional space are when

1) the vector $\underline{k}(p)$ moves in space in such a way that its component $\underline{k}_{ij}$ on the plane defined by the position vectors $\underline{r}_i, \underline{r}_j$ has a constant direction, i.e. the angles between $\underline{k}_{ij}$ and $\underline{r}_i, \underline{r}_j$ are independent of p.
2) $\underline{r}_i = c\underline{r}_j, \ c \in \mathcal{R}$

The second case clearly arises only when the array is a linear one.

The first case can only arise in the case of planar arrays and not if the array elements are placed in 3-dimensional space. To see why this is the case, let us consider a 3-dimensional array, that is an array where there are at least 3 element position vectors $\underline{r}_i, \underline{r}_j, \underline{r}_k$ such that they are not all on the same plane. Condition (52) requires that the the angles between $\underline{k}_{ij}$ and $\underline{r}_i, \underline{r}_j$ are constant. However, the same has to be true for the projection $\underline{k}_{ik}$ of $\underline{u}(p)$ on the plane spanned by $\underline{r}_i, \underline{r}_k$, which is by assumption different by the plane spanned by $\underline{r}_i, \underline{r}_j$. These two restrictions, along with the assumption that $\|\underline{k}(p)\| = \pi$ imply that $\underline{u}(p)$ is constant, independent of p.

■

VI. Conclusions

The issues of existence and uniqueness of hyperhelical manifold curves were addressed in this paper. Initially, Eq. (11) which provides the natural representation of hyperhelical manifold curves embedded in $\mathcal{C}^N$ is examined and proved. This implies that all hyperhelical manifold curves lie on a complex sphere. Next, hyperhelical manifold curves describing array systems were considered. The requirement that the phase of the array manifold is real led us to restrict ourselves to a subset of all the possible hyperhelical manifold curves in $\mathcal{C}^N$, namely those for which Eq. (45) is true. Based on this, Theorem 2 asserts that the general parametric representation of hyperhelical array manifold curves is given by Eq. (40). Finally, in Theorem 3 it is shown that only linear and planar arrays can be described by hyperhelical array manifold curves. Thus, existing algorithms taking advantage of the hyperhelical structure of the array manifold curve cannot be applied to 3-dimensional arrays, no matter how the directional parameters are represented.

Appendix
Proof of the curvatures formula of Theorem 1

We will begin with the proof of Eq. (53) and the method of 2-step mathematical induction will be used to prove the desired expression.

Proof:

Based on the results for the first and second order coordinate vectors and curvatures, the emerging pattern of the formulae leads to the the following expression for the i-th coordinate vector:

$$\begin{aligned}\underline{u}_i = &\frac{j^i}{\kappa_1\kappa_2\ldots\kappa_{i-1}} \cdot \\ &\cdot \left[\left(\sum_{n=1}^{\left\lfloor\frac{i-1}{2}\right\rfloor+1}(-1)^{n-1} \ b_{i-1,n} \ \underline{\tilde{r}}^{i-2n+2}\right)\odot \underline{\text{a}}\right]\end{aligned} \quad (53)$$

Eq. (53) is valid for $i = 1, \ldots, d$.

1) $i = 2$ and $i = 3$: If we apply formula (53) for $i = 2$ and $i = 3$ it is straightforward to show that it is verified.

2) $i = k-1$, $i = k$: Let us assume that this formula is correct for $i = k-1$ and $i = k$, that is

$$\begin{aligned}\underline{u}_{k-1} = &\frac{j^{k-1}}{\kappa_1\kappa_2\cdots\kappa_{k-2}} \cdot \\ &\cdot \left[\left(\sum_{n=1}^{\left\lfloor\frac{k-2}{2}\right\rfloor+1}(-1)^{n-1}\cdot b_{k-2,n}\cdot\underline{\tilde{r}}^{k+1-2n}\right)\odot\underline{\text{a}}\right]\end{aligned}$$

and

$$\begin{aligned}\underline{u}_k = &\frac{j^k}{\kappa_1\cdot\kappa_2\cdots\kappa_{k-1}} \cdot \\ &\cdot \left[\left(\sum_{n=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor+1}(-1)^{n-1}\cdot b_{k-1,n}\cdot\underline{\tilde{r}}^{k+2-2n}\right)\odot\underline{\text{a}}\right]\end{aligned}$$

Then

$$\begin{aligned}\underline{u}_k^{'} = &\frac{j^{k+1}}{\kappa_1\cdot\kappa_2\cdots\kappa_{k-1}} \cdot \\ &\cdot \left[\left(\sum_{n=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor+1}(-1)^{n-1}\cdot b_{k-1,n}\cdot\underline{\tilde{r}}^{k+3-2n}\right)\odot\underline{\text{a}}\right]\end{aligned}$$

and

$$\begin{aligned}\kappa_{k-1}\cdot\underline{u}_{k-1} = &\frac{j^{k-1}\cdot\kappa_{k-1}^2}{\kappa_1\cdots\kappa_{k-1}} \cdot \\ &\cdot \left[\left(\sum_{n=1}^{\left\lfloor\frac{k-2}{2}\right\rfloor+1}(-1)^{n-1}b_{k-2,n}\cdot\underline{\tilde{r}}^{k+1-2n}\right)\odot\underline{\text{a}}\right]\end{aligned}$$

By setting $m \triangleq n+1$

$$\begin{aligned}\kappa_{k-1}\cdot\underline{u}_{k-1} = &\frac{-j^{k+1}\cdot\kappa_{k-1}^2}{\kappa_1\cdots\kappa_{k-1}} \cdot \\ &\cdot \left[\left(\sum_{m=2}^{\left\lfloor\frac{k}{2}\right\rfloor+1}(-1)^{m-2}b_{k-2,m-1}\cdot\underline{\tilde{r}}^{k+3-2m}\right)\odot\underline{\text{a}}\right]\end{aligned}$$

Substituting again $n \triangleq m$, we get

$$k_{k-1} \cdot \underline{u}_{k-1} = \frac{j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdots \kappa_{k-1}} \cdot \cdot \left[\left(\sum_{n=2}^{\lfloor \frac{k}{2} \rfloor + 1} (-1)^{n-1} b_{k-2,n-1} \cdot \underline{\tilde{r}}^{k+3-2n} \right) \odot \underline{\mathrm{a}} \right]$$

3) $i = k+1$:

$$\begin{aligned} \underline{u}_{k+1} &= \frac{\underline{u}_k' + \kappa_{k-1} \cdot \underline{u}_{k-1}}{\kappa_k} = \frac{j^{k+1}}{\kappa_1 \cdots \kappa_k} \cdot \\ &\cdot \left[\left(\sum_{n=1}^{\lfloor \frac{k-1}{2} \rfloor] + 1} (-1)^{n-1} b_{k-1,n} \cdot \underline{\tilde{r}}^{k+3-2n} \right) \odot \underline{\mathrm{a}} \right] \\ &+ \frac{j^{k+1} \cdot \kappa_{k-1}^2}{\kappa_1 \cdots \kappa_k} \cdot \\ &\cdot \left[\left(\sum_{n=2}^{\lfloor \frac{k}{2} \rfloor + 1} (-1)^{n-1} b_{k-2,n-1} \cdot \underline{\tilde{r}}^{k+3-2n} \right) \odot \underline{\mathrm{a}} \right] \end{aligned} \tag{54}$$

The formula of Eq. (54) is not the same as Eq. (53), which we are trying to prove . From the first sum of the right hand, the term for $n = \lfloor \frac{k}{2} \rfloor + 1$ is missing, while from the second sum, the term for $n = 1$ is missing. We will show now, that both these terms equal to 0, or equivalently, that the corresponding coefficients $b_{i,n} = 0$.

The coefficients $b_{i,n}$ are 0 if one of the following conditions is satisfied:

1) $i = n \wedge i > 2$
2) $i < n$
3) $n = 0$

Therefore, the second missing term can easily be proven to be equal to 0, since

$$b_{k-2,n-1} \overset{n=1}{=} b_{k-2,0} = 0$$

If $k = 2\rho+1$, $\rho = 1, 2 \ldots$, then $\lfloor \frac{k}{2} \rfloor + 1 = \rho + 1 = \lfloor \frac{k-1}{2} \rfloor + 1$ and the upper limit in the first sum is $\lfloor \frac{k}{2} \rfloor + 1$ as it should be. Therefore, there is no need to show that the term under consideration is equal to 0.

If $k = 2\rho$, $\rho = 1, 2 \ldots$ then the term under consideration is

$$n = \left\lfloor \frac{k}{2} \right\rfloor + 1 = \rho + 1$$

Therefore, we have to show that $b_{2\rho-1,\rho+1} = 0$.

The coefficients $b_{i,n}$ can be defined recursively, as can be seen in Eq. (15). In every step of the recursion, the coefficient can be expressed as the sum of two terms, in each one of which there is another coefficient, with different indices. In every step, in both of the emerging coefficients, the difference between the first and the second index is decreased by one. However, in the second term, the first index is decreased each time by two and the second by one, while in the first term only the first index is decreased by one. In order for $b_{2\rho-1,\rho+1} = 0$ to be zero, all of the emerging coefficients in this recursion must be also zero. Equivalently, the indices of all the emerging coefficients must satisfy one of the conditions mentioned above.

We can represent this recursion as a binary tree, each node of which will represent a different coefficient. For the root of this tree to be 0, all the leafs have to be zero too. However, if we prove that the coefficient emerging always from the second term of the recursive equation (15), that is the rightmost leaf of the tree is equal to 0 then, all the other coefficients will be equal to 0 as well, because even though the difference in the indexes is decreased by the same amount in each recursion (each level of the tree), the indices of the rightmost coefficients (the nodes in the path from the root to the rightmost coefficient) approach the limit $i = 2$ faster than the nodes in any other path.

After x steps of the recursion, the indices of the rightmost leaf will be

$$b_{i_x,n_x} = b_{2\rho-1-2x,\rho+1-x}$$

The indices have to be equal, in order for the coefficient to be zero, so

$$2\rho - 1 - 2x = \rho + 1 - x \Rightarrow x = \rho - 2$$

Therefore, after $x = \rho - 2$ steps of the recursion, indices will be

$$\begin{aligned} i_x &= 2\rho - 1 - 2(\rho - 2) = 3 > 2 \\ n_x &= \rho + 1 - (\rho - 2) = 3 > 2 \end{aligned}$$

Thus, it has been proven, that all the coefficients for this term are 0 and the proof for the coordinate vectors is complete. The proof for the curvatures is straightforward using Eq. (14), since this formula is based on the formula of the coordinate vectors, but for the shake of brevity, only a short proof for the independence of the arc length will be presented next, which is what is required for the curve $\mathcal{A}$ to be a hyperhelix.

By definition

$$\begin{aligned} \kappa_i^2(s) &= \left\| \underline{u}_i'(s) + \kappa_{i-1}(s) \underline{u}_{i-1}(s) \right\|^2 = \\ &\underline{u}_i'(s)^H \underline{u}_i'(s) + \kappa_{i-1}^2(s) \underline{u}_{i-1}^H(s) \underline{u}_{i-1}(s) + \\ &+ 2\kappa_{i-1}(s) \mathrm{Re} \left\{ \underline{u}_{i-1}^H(s) \underline{u}_i'(s) \right\} \end{aligned} \tag{55}$$

However from Eq. (53) it is known that the i-th coordinate vector $\underline{u}_i(s)$ can be written as

$$\underline{u}_i(s) = \underline{x}_i \odot \underline{\mathrm{a}}(s)$$

where

$$\underline{x}_i \triangleq \frac{j^i}{\kappa_1 \kappa_2 \ldots \kappa_{i-1}} \cdot \left(\sum_{n=1}^{\lfloor \frac{i-1}{2} \rfloor + 1} (-1)^{n-1} \, b_{i-1,n} \, \underline{\tilde{r}}^{i-2n+2} \right)$$

is independent of s. Based on this and on the fact that by assumption $\underline{\mathrm{a}}'(s) = -j\underline{\tilde{r}} \odot \underline{\mathrm{a}}(s)$, then all the terms of Eq. (55) are independent of s provided that $\kappa_{i-1}(s)$ is. However, κ_1, κ_2 are independent of s , as can be seen in Eq. (12) and by induction all the consequent are as well.

■

References

[1] R. Schmidt, "A signal subspace approach to multiple emitter location and spectral estimation," *PHD Dissertation, Stanford University*, 1982.

[2] M. Pesavento, A. Gershman, and M. Haardt, "Unitary root-music with a real-valued eigendecomposition: a theoretical and experimental performance study," *IEEE Transactions on Signal Processing*, vol. 48, no. 5, pp. 1306–1314, May 2000.

[3] R. Lorenz and S. Boyd, "Robust minimum variance beamforming," *IEEE Transactions on Signal Processing*, vol. 53, no. 5, pp. 1684–1696, May 2005.

[4] S. Yan and J. Hovem, "Array pattern synthesis with robustness against manifold vectors uncertainty," *Oceanic Engineering, IEEE Journal of*, vol. 33, no. 4, pp. 405–413, Oct. 2008.

[5] J. Mosher and R. Leahy, "Source localization using recursively applied and projected (rap) music," *IEEE Transactions on Signal Processing*, vol. 47, no. 2, pp. 332–340, Feb 1999.

[6] M. Buhren, M. Pesavento, and J. Bohme, "Virtual array design for array interpolation using differential geometry," in *IEEE International Conference on Acoustics, Speech, and Signal Processing, 2004. (ICASSP '04).*, vol. 2, May 2004, pp. ii–229–32.

[7] M. Gavish and A. Weiss, "Array geometry for ambiguity resolution in direction finding," *IEEE Transactions on Antennas and Propagation*, vol. 44, no. 6, pp. 889–895, Jun 1996.

[8] A. Manikas and C. Proukakis, "Modeling and estimation of ambiguities in linear arrays," *IEEE Transactions on Signal Processing*, vol. 46, no. 8, pp. 2166–2179, 1998.

[9] A. Manikas, H. R. Karimi, and I. Dacos, "Study of the detection and resolution capabilities of a one-dimensional array of sensors by using differential geometry," *IEE Proceedings - Radar, Sonar and Navigation*, vol. 141, no. 2, pp. 83–92, 1994.

[10] G. Efstathopoulos and A. Manikas, "Extended array manifolds: Functions of array manifolds," *IEEE Transactions on Signal Processing*, vol. 59, no. 7, pp. 3272–3287, 2011.

[11] I. Dacos and M. A., "Estimating the manifold parameters of one - dimensional array of sensors," *J. Franklin Inst. (Eng. Appl. Math.)*, vol. 332B, no. 3, pp. 307–332, 1995.

[12] N. J. Malloy, "Array manifold geometry and sparse volumetric array design optimization," in *Conference Record of the Forty-First Asilomar conference on Signals, Systems and Computers, 2007*, November 2007, pp. 1257–1261.

[13] U. Baysal and R. L. Moses, "On the geometry of isotropic arrays," *IEEE Transactions on Signal Processing*, vol. 51, no. 6, pp. 1469–1478, 2003.

[14] D. Asztely, B. Ottersten, and A. Swindlehurst, "Generalised array manifold model for wireless communication channels with local scattering," *IEE Proceedings - Radar, Sonar and Navigation,*, vol. 145, no. 1, pp. 51–57, Feb 1998.

[15] M. A. Doron, M. A. Doron, and E. Doron, "Wavefield modeling and array processing. III. Resolution capacity," *IEEE Transactions on Signal Processing*, vol. 42, no. 10, pp. 2571–2580, 1994.

[16] ——, "Wavefield modeling and array processing. II. Algorithms," *IEEE Transactions on Signal Processing*, vol. 42, no. 10, pp. 2560–2570, 1994.

[17] ——, "Wavefield modeling and array processing. I. Spatial sampling," *IEEE Transactions on Signal Processing*, vol. 42, no. 10, pp. 2549–2559, 1994.

[18] A. Manikas, *Differential Geometry in Array Processing*. Imperial College Press, 2004.

[19] U. Guggenheimer, *Differential Geometry*. McGraw Hill, 1963.

[20] N. Dowlut and A. Manikas, "A polynomial rooting approach to super-resolution array design," *IEEE Transactions on Signal Processing*, vol. 48, no. 6, pp. 1559–1569, 2000.

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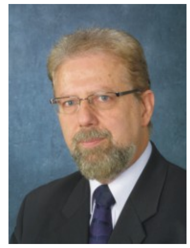

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