Investigating Hyperhelical Array Manifold Curves Using the Complex Cartan Matrix

Athanassios Manikas, *Senior Member, IEEE*, Harry Commin, *Student Member, IEEE*, and Adham Sleiman

***Abstract*—The differential geometry of array manifold curves has been investigated extensively in the literature, leading to numerous applications. However, the existing differential geometric framework restricts the Cartan matrix to be purely real and so the vectors of the moving frame $\mathbb{U}(s)$ are found to be orthogonal only in the wide sense (i.e. only the real part of their inner product is equal to zero). Imaginary components are then accounted for separately using the concept of the inclination angle of the manifold.**

The purpose of this paper is therefore to present an alternative theoretical framework which allows the manifold curve in $\mathfrak{C}^N$ to be characterised in a more convenient and direct manner. A continuously differentiable strictly orthonormal basis is established and forms a platform for deriving a generalised complex Cartan matrix with similar properties to those established under the previous framework. Concepts such as the radius of circular approximation, the manifold curve radii vector and the frame matrix are also revisited and rederived under this new framework.

***Index Terms*—Array manifold, differential geometry, array processing.**

NOTATION

a, A	Scalar
$\underline{a}, \underline{A}$	Column vector
$\mathbb{A}$	Matrix
$(\cdot)^T$, $(\cdot)^H$	Transpose, conjugate transpose
$\lvert\cdot\rvert$	Absolute value
$\lVert\cdot\rVert$	Euclidean norm of vector
$\underline{a}^b$	Element-by-element power
$\exp(\underline{a})$	Element-by-element exponential
$\mathrm{diag}\{\underline{a}\}$	Diagonal matrix whose diagonal entries are $\underline{a}$
$\mathbb{I}_N$	$(N \times N)$ identity matrix
$\underline{0}_N$	$(N \times 1)$ vector of zeros
$\mathrm{Tr}\{\cdot\}$	Matrix trace operator
$\odot$	Hadamard (element-by-element) product
$\mathfrak{R}$	The set of real numbers
$\mathfrak{C}$	The set of complex numbers

I. INTRODUCTION

CONSIDER an array of N sensors (residing in three-dimensional real space) receiving one or more narrowband plane waves. The response of the array to a signal incident from azimuth $\theta \in [0, 360°)$ and elevation $\phi \in (-90, 90°)$ is commonly modelled using the $(N \times 1)$ complex array response vector, or manifold vector:

$$\underline{a}(\theta, \phi) \triangleq \exp\left(-j\left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right]\underline{k}(\theta, \phi)\right) \quad (1)$$

The authors are with the Department of Electrical and Electronic Engineering, Imperial College London, South Kensington Campus, London, SW7 2AZ, UK e-mail: (a.manikas@imperial.ac.uk).

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where the $(N \times 3)$ real matrix $[\underline{r}_x, \underline{r}_y, \underline{r}_z]$ denotes the array sensor locations:

$$[\underline{r}_x, \underline{r}_y, \underline{r}_z] = [\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N]^T \in \mathcal{R}^{N\times 3} \quad (2)$$

For the purposes of this paper, it is convenient to define the phase reference at the centroid of the array. In Equation 1, $\underline{k}(\theta, \phi)$ is the wavenumber vector:

$$\underline{k}(\theta, \phi) \triangleq \frac{2\pi}{\lambda}\overbrace{[\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi)]}^{\underline{u}(\theta,\phi)} \quad (3)$$

where λ is the carrier signal wavelength and $\underline{u}(\theta, \phi)$ is the (3×1) real unit vector pointing from (θ, ϕ) towards the origin.

As is common practice, the notation used for $\underline{a}(\theta, \phi)$ in Equation 1 ignores dependence upon the constant, known quantities $\left[\underline{r}_x, \underline{r}_y, \underline{r}_z\right]$ and λ. However, it is important to note that the manifold vector is by no means restricted to a (θ, ϕ)-parameterisation. Firstly, this is because the (θ, ϕ)-parameterisation is not unique; other valid directional parameterisations exist (with cone-angle parameterisation particularly useful for the analysis of planar arrays [1]). Secondly, many other variable parameters of interest (besides direction) can be incorporated into the response vector modelling. When these additional parameters (such as delay, Doppler, polarisation, subcarrier and spreading/scrambling codes) are included, the resulting response vector is referred to as an *extended manifold vector*, the properties of which have been investigated in [2].

Therefore, for the sake of generality, the response vector will simply be denoted as $\underline{a}(\underline{p})$, where the vector $\underline{p}$ may comprise any or all of the aforementioned variable parameters (and/or others). The central purpose of this paper then lies in exploring the geometrical nature of the mathematical object which is traced out by $\underline{a}(\underline{p})$ when $\underline{p}$ is evaluated across the range of all feasible parameter values (denoted by the parameter space, Ω). This resulting object is called the *array manifold*. It completely characterises the array system and is formally defined as:

$$\mathcal{A} \triangleq \left\{\underline{a}(\underline{p}), \quad \forall \underline{p} \in \Omega\right\} \quad (4)$$

In the single-parameter case (i.e. $\underline{p} = p$), it can be seen that $\underline{a}(p)$ traces out a *curve* in $\mathfrak{C}^N$. For two parameters, the manifold is a *surface* and, similarly, for larger numbers of parameters, the manifold is some higher dimensional object.

A branch of mathematics dedicated to the investigation of these kinds of (differentiable) manifolds is differential geometry [3,4]. The tools of differential geometry have already been applied extensively in the array signal processing literature [5]. The main body of this existing research relates to the study of manifold curves and surfaces. Although the geometrical properties of array manifold surfaces have been investigated directly [6], this is a more complicated task than for curves. To simplify matters, it has been shown that array manifold surfaces can alternatively be considered to consist of families of constant-parameter curves [7]. Therefore, the theoretical framework introduced in this paper will be done so with a focus on array manifold curves.

A particularly important class of array manifold curves is the hyperhelix. Hyperhelices are especially convenient to analyse since all their curvatures are constant (do not vary from point to point) and may be calculated recursively. Although all linear arrays of isotropic sensors have hyperhelical manifold curves, the useful properties of the hyperhelix have also been exploited for the analysis of planar arrays using cone-angle parameterisation and the concept of the "equivalent linear array" [5]. Furthermore, hyperhelical manifold curves have been identified in the analysis of extended array manifolds [2]. Therefore, in this paper, particularly close attention will be paid to the analysis of hyperhelical manifold curves.

In order to highlight the motivation of this paper, it is useful at this point to briefly review the existing differential geometric framework that has been applied to the analysis of array manifold curves. A short summary of some of its key applications in the literature will then be given.

A. Geometry of Array Manifold Curves: Traditional Approach

Given a curve $\underline{a}(p) \in \mathfrak{C}^N$ as a function of a single real parameter $p \in \mathfrak{R}$ (such as azimuth, θ, or elevation, ϕ), the arc length along the manifold curve, $s(p)$, and its rate of change, $\dot{s}(p)$, are defined, respectively, as:

$$s(p) \triangleq \int_0^p \dot{s}(p_0) dp_0 \tag{5}$$

$$\dot{s}(p) \triangleq \frac{ds}{dp} = \|\dot{\underline{a}}(p)\| \tag{6}$$

A useful feature of parameterising a manifold curve in terms of arc length, s, is that it is an *invariant* parameter. This means that the tangent vector to the curve (expressed in terms of s):

$$\underline{a}'(s) \triangleq \frac{d}{ds}\underline{a}(s) = \frac{dp}{ds}\frac{d}{dp}\underline{a}(p) = \frac{\dot{\underline{a}}(p)}{\|\dot{\underline{a}}(p)\|} \tag{7}$$

is always unit length. (Note that differentiation with respect to s has been denoted by "prime" and differentiation with respect to p by "dot". This convention will be followed throughout this paper).

The tangent vector $\underline{a}'(s_0)$ provides useful local information about the manifold curve in the neighbourhood of s_0 (it is a geometric approximation of the first order). However, in order to build up a full (local) characterisation of the manifold curve, it is necessary to attach some additional vectors to the running point, s_0. Specifically, an orthonormal *moving frame* of (local) coordinate vectors can be constructed. Then, the manner in which this frame twists and turns as the running point progresses along the curve will be shown to provide a profoundly meaningful description of the curve. However, the way in which these frame vectors are chosen is of central importance to this paper. A subtly different approach will be seen to provide significant new insight.

Under the traditional approach, the manifold vector $\underline{a}(p)$, residing in $\mathfrak{C}^N$, is instead considered to comprise $2N$ *real* components. For this reason, the moving frame, $\mathbb{U}_{\mathrm{w}}(s)$, consists of up to $2N$ complex vectors:

$$\mathbb{U}_{\mathrm{w}}(s) \triangleq \left[\underline{u}_{\mathrm{w},1}(s), \underline{u}_{\mathrm{w},2}(s), \ldots, \underline{u}_{\mathrm{w},d}(s)\right] \tag{8}$$

where $N-1 \leq d \leq 2N$ (depending on the level of symmetricity exhibited by the sensor array). Together with $\mathbb{U}_{\mathrm{w}}(s)$, up to $2N-1$ non-zero real curvatures can be defined. These allow the motion of the moving frame to be expressed as:

$$\mathbb{U}'_{\mathrm{w}}(s) = \mathbb{U}_{\mathrm{w}}(s)\mathbb{C}_{\mathrm{r}}(s) \tag{9}$$

where $\mathbb{C}_{\mathrm{r}}(s)$ is the purely *real* Cartan matrix, which contains the curvatures, κ_i, according to the following skew-symmetric structure:

$$\mathbb{C}_{\mathrm{r}}(s) = \begin{bmatrix} 0 & -\kappa_1 & 0 & \cdots & 0 \\ \kappa_1 & 0 & -\kappa_2 & \ddots & \vdots \\ 0 & \kappa_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\kappa_{d-1} \\ 0 & \cdots & 0 & \kappa_{d-1} & 0 \end{bmatrix} \tag{10}$$

As a consequence of constraining the entries of $\mathbb{C}_{\mathrm{r}}(s)$ to be purely real, it is found that the moving frame, $\mathbb{U}_{\mathrm{w}}(s)$, can only be orthonormal in the "wide sense" (i.e. applying only to the *real* part):

$$\mathrm{Re}\left\{\mathbb{U}_{\mathrm{w}}^H(s)\mathbb{U}_{\mathrm{w}}(s)\right\} = \mathbb{I}_d \tag{11}$$

As a result, it has been proven [5, Eq. 2.23] that this leads to $\mathbb{C}_{\mathrm{r}}(s)$ taking the form:

$$\mathbb{C}_{\mathrm{r}}(s) = \mathrm{Re}\left\{\mathbb{U}_{\mathrm{w}}^H(s)\mathbb{U}'_{\mathrm{w}}(s)\right\} \tag{12}$$

This means that $\mathbb{C}_{\mathrm{r}}(s)$ does not, in general, uniquely describe the shape and size of the manifold curve (since any imaginary components are ignored). Only in the special case of arrays with sensors located symmetrically about the origin is the manifold seen to admit a representation entirely in $\mathfrak{R}^N$. In the general case, therefore, imaginary components must be accounted for separately, using the concept of the *inclination angle* of the manifold (defined as the angle between the manifold vector and a certain subset of the even-numbered frame vectors [5, p.45]).

The purpose of the new theoretical framework presented in this paper is to allow the manifold curve in $\mathfrak{C}^N$ to be characterised in a more convenient and direct manner. The key to doing this lies in reformulating Equations 8 - 12. In particular, a continuously differentiable *strictly orthonormal* basis, $\mathbb{U}(s) \in \mathfrak{C}^{N \times N}$, is established and forms a basis for deriving a generalised complex Cartan matrix, $\mathbb{C}(s)$, with similar properties to those established under the previous framework. Concepts such as the radius of circular approximation, the manifold curve radii vector and the frame matrix are also revisited and rederived under this new framework.

B. Exemplary Results and Motivation

The usefulness and importance of applying the tools of differential geometry to the analysis of the array manifold can be demonstrated using a number of results found in the literature. In [8], the fundamental performance capabilities (detection, resolution and estimation error) of an array system were defined in the context of the array manifold and derived explicitly using the circular approximation of the array manifold. The way in which anomalies in the manifold lead to array ambiguities was investigated in [9]. A differential geometric perspective was applied to the topic of array design (i.e. the judicious placement of array sensors) in [10]. The sensitivity of array systems to uncertainties in sensor locations, based on their location in the overall array geometry, was analysed in [11]. In [12], the task of designing virtual arrays for the purpose of array interpolation was explored.

The study of more sophisticated array systems which incorporate additional system and channel parameters (such as code division multiple access spreading/scrambling codes, lack of synchronisation, Doppler effects, polarisation parameters and subcarriers) has been introduced using the concept of extended array manifolds in [2].

C. Paper Organisation

The remainder of this paper is organised as follows. In Section II, the basic principles of the proposed theoretical framework are presented. The strictly orthogonal moving frame is established and the complex Cartan matrix is derived. In Section III, specific properties of hyperhelical manifold curves are investigated. Explicit expressions are derived for the orthonormal frame vectors and entries of the complex Cartan matrix and numerical examples are presented. Finally, the paper is concluded in Section IV.

II. Curves in $\mathfrak{C}^N$

In this section, the strictly orthonormal moving frame (or moving polyhedron), $\mathbb{U}(s)$, will be formulated. Using the definition of $\mathbb{U}(s)$, the exact structure of the complex Cartan matrix, $\mathbb{C}(s)$, will then be derived.

A. The Moving Polyhedron

In order to describe meaningfully the way in which the curve $\underline{a}(s)$ twists through N-dimensional complex space (at a point s), the *moving frame* of N orthonormal coordinate vectors, $\mathbb{U}(s)$, at the running point, s:

$$\mathbb{U}(s) \triangleq [\underline{u}_1(s), \underline{u}_2(s), \ldots, \underline{u}_N(s)] \tag{13}$$

will now be derived, where strict orthonormality implies:

$$\mathbb{U}^H(s)\mathbb{U}(s) = \mathbb{I}_N \tag{14}$$

Specifically, in accordance with common practice in $\mathfrak{R}^N$ (see, for example, [3, p.159] and [4, p.13]), the unit tangent vector is first defined as:

$$\underline{u}_1(s) \triangleq \underline{a}'(s) \tag{15}$$

and then all subsequent coordinate vectors (for $k \geq 2$) are obtained via Gram-Schmidt orthonormalisation:

$$\underline{u}_k(s) = \frac{\underline{\mathrm{v}}_k(s)}{\|\underline{\mathrm{v}}_k(s)\|} \tag{16a}$$

$$\underline{\mathrm{v}}_k(s) \triangleq \underline{u}'_{k-1}(s) - \sum_{m=1}^{k-1} \left(\underline{u}_m^H(s)\underline{u}'_{k-1}(s)\right) \underline{u}_m(s) \tag{16b}$$

In other words, each successive coordinate vector simply points in the direction that the hyperplane spanned by the previously-defined vectors is moving.

It is worth noting that, under certain circumstances, the array manifold curve may not span the full N-dimensional complex observation space and so it is not always necessary to define N frame vectors. Consider, for example, array configurations where a sensor exists at the array centroid (i.e. the phase origin). It is clear that that sensor's response must be constant for all p and, consequently, the p-curve cannot depart from the $(N-1)$-dimensional hyperplane associated with this constant value.

B. The Complex Cartan Matrix

The Cartan matrix provides the curvatures of the curve $\underline{a}(s)$, which in turn are the curvilinear coordinates of the curve on the moving polyhedron $\mathbb{U}(s)$. These curvatures are a means of assessing the changes in the polyhedron, expressed by the first derivative, $\mathbb{U}'(s)$, of $\mathbb{U}(s)$, and hence the Cartan matrix, $\mathbb{C}(s)$, is used to express $\mathbb{U}'(s)$ as a function of $\mathbb{U}(s)$:

$$\mathbb{U}'(s) = \mathbb{U}(s)\mathbb{C}(s) \tag{17}$$

Note that in complex multi-dimensional space, $\mathbb{U}(s) \in \mathfrak{C}^{N \times N}$ and hence $\mathbb{C}(s) \in \mathfrak{C}^{N \times N}$. In particular, $\mathbb{C}(s)$ should be skew-Hermitian since this would yield, as a special case, the familiar skew-symmetric purely real Cartan matrix for curves in $\mathfrak{R}^N$.

It is proven in Appendix A that the complex Cartan matrix can be written in the form:

$$\mathbb{C}(s) = \begin{bmatrix} \underline{u}_1^H\underline{u}'_1 & -\|\underline{\mathrm{v}}_2\| & 0 & \cdots & 0 \\ \|\underline{\mathrm{v}}_2\| & \underline{u}_2^H\underline{u}'_2 & -\|\underline{\mathrm{v}}_3\| & \ddots & \vdots \\ 0 & \|\underline{\mathrm{v}}_3\| & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\|\underline{\mathrm{v}}_N\| \\ 0 & \cdots & 0 & \|\underline{\mathrm{v}}_N\| & \underline{u}_N^H\underline{u}'_N \end{bmatrix} \tag{18}$$

which is indeed skew-Hermitian:

$$\mathbb{C}^H(s) = -\mathbb{C}(s) \tag{19}$$

This property is easily proven, since differentiating the identity $\underline{u}_k^H\underline{u}_k = 1$ leads to:

$$\begin{aligned} &\underline{u}_k^H\underline{u}'_k + \underline{u}'^H_k\underline{u}_k = 0 \\ &\Rightarrow \mathrm{Re}\left\{\underline{u}_k^H\underline{u}'_k\right\} = 0 \end{aligned} \tag{20}$$

and so the diagonal entries of $\mathbb{C}(s)$ are purely imaginary.

Clearly, the structure of $\mathbb{C}(s)$ (Equation 18) shows certain similarities to $\mathbb{C}_\mathrm{r}(s)$ in the previous framework (Equation 10). The key differences to note are that, firstly, $\mathbb{C}(s)$ is generally a smaller matrix ($(N \times N)$ compared to $(d \times d)$) and, secondly, it can have non-zero, albeit purely imaginary, diagonal entries. As a result, in the general case ($2 \leq k \leq N-1$), the frame vector derivative $\underline{u}_k'(s)$ is a linear combination of not only $\underline{u}_{k-1}(s)$ and $\underline{u}_{k+1}(s)$ (with purely real coefficients) but also $\underline{u}_k(s)$ itself (with the purely imaginary coefficient $\underline{u}_k^H(s)\underline{u}_k'(s)$). In this way, $\mathbb{C}(s)$ completely characterises the fully complex nature of the curve (including the concept of inclination angle in the previous framework). This point can now be demonstrated by considering the radius of the circular approximation of the manifold.

C. The Radius of the Circular Approximation

A powerful tool in the analysis of array manifold curves is the circular approximation of the manifold [5, Ch.8]. More specifically, it has been proven that any sufficiently small section (arc length) of an array manifold curve can be approximated as a circular arc. Using the previous differential geometric framework, it was shown that the inverse of the radius of the circular approximation was only equal to the principal curvature, κ_1, in the special case of symmetric linear arrays. For general array geometries, however, the inclination angle of the manifold, ζ, needed to be accounted for as follows:

$$\begin{aligned} \hat{\kappa}_1(s) &= \kappa_1(s)\sin\zeta(s) && (21)\\ &= \sqrt{\left\|\underline{u}_1'(s)\right\|^2 - \left|\underline{u}_1^H(s)\underline{u}_1'(s)\right|^2} && (22) \end{aligned}$$

the proof of which was derived in [5, p.209].

The corresponding curvature in the complex Cartan matrix derived in this paper is $\|\underline{\mathrm{v}}_2(s)\|$:

$$\begin{aligned} \|\underline{\mathrm{v}}_2(s)\| &= \left\|\underline{u}_1'(s) - \left(\underline{u}_1^H(s)\underline{u}_1'(s)\right)\underline{u}_1(s)\right\| \\ &= \sqrt{\left\|\underline{u}_1'(s)\right\|^2 - \left|\underline{u}_1^H(s)\underline{u}_1'(s)\right|^2} && (23) \end{aligned}$$

which provides the desired result directly. This demonstrates both the equivalence between the two theoretical frameworks as well as the directness and convenience of using the complex Cartan matrix.

III. Hyperhelical Array Manifold Curves

Having established (in Equations 16 and 18) the new framework for investigating the geometry of array manifold curves, the important class of curves which have hyperhelical shape can now be studied in detail. In this way, a number of concepts such as the frame matrix and manifold radii vector can be rederived in order to compare to the analogous quantities obtained via the previous framework. New results regarding the eigendecomposition of the complex Cartan matrix are also given, followed by specific numerical examples.

Consider an array system with a response vector having the form:

$$\underline{a}(p) = \exp\left(-j\pi\underline{r}\cos p + \underline{w}\right) \quad (24)$$

where $\underline{r}$ and $\underline{w}$ are constant vectors and p is the variable parameter of interest. The most commonly seen example of such a response vector is that of of a linear array (where $\underline{r} = \underline{r}_x\cos(\phi_0)$ and $\underline{w} = \underline{0}_N$) as a function of azimuth ($p = \theta$). However, this model also applies more generally to ϕ, α and β curves of planar arrays (where α and β denote cone angles), via the concept of the equivalent linear array [5, p.113]. In such cases, $\underline{r} \in \Re^N$ denotes the equivalent linear array sensor locations.

It will be shown in this section that the manifold curve traced out by any $\underline{a}(p)$ with this structure (Equation 24) is of hyperhelical shape. A number of important properties of hyperhelical manifold curves will then be derived explicitly.

A. Manifold Vector Derivatives

As a preliminary to investigating the differential geometry of the hyperhelical array manifold, a compact differentiation procedure of the response vector is established. Applying the chain rule, leads to:

$$\begin{aligned} \underline{a}'(s) &= \frac{\dot{\underline{a}}(p)}{\dot{s}(p)} = \frac{\dot{\underline{a}}(p)}{\|\dot{\underline{a}}(p)\|} \\ &= \frac{j\pi\underline{r}\sin p \odot \underline{a}(p)}{\pi\|\underline{r}\|\sin p} \\ &= j\overline{\underline{r}} \odot \underline{a}(p) && (25) \end{aligned}$$

where the normalised (equivalent) sensor locations have been defined as:

$$\overline{\underline{r}} \triangleq \frac{\underline{r}}{\|\underline{r}\|} \quad (26)$$

Applying the chain rule repeatedly to $\underline{a}(s)$ in this way leads a pattern to emerge which is straightforward to prove by induction:

$$\underline{a}^{(k)}(s) \triangleq \frac{d^k\underline{a}(s)}{ds^k} = \left(j\overline{\underline{r}}\right)^k \odot \underline{a}(s) \quad (27)$$

For notational convenience, we define the diagonal matrix $\mathbb{R}$:

$$\mathbb{R} \triangleq \operatorname{diag}(\overline{\underline{r}}) \in \Re^{N\times N} \quad (28)$$

which allows Equation 27 to be equivalently written as:

$$\underline{a}^{(k)}(s) = j^k\mathbb{R}^k\underline{a}(s) \quad (29)$$

This notation will prove useful in investigating the differential geometry of the hyperhelical array manifold curve. Note that $\operatorname{Tr}\left\{\mathbb{R}^2\right\} = \|\overline{\underline{r}}\|^2 = 1$.

B. Differential Geometry of Hyperhelical Manifold Curves

It is now possible to investigate the orthonormal basis $\mathbb{U}(s)$ and the elements of the Cartan matrix, $\mathbb{C}(s)$, for the hyperhelical array manifold. Indeed, in Appendix B, it is shown that the (unnormalised) orthogonal frame vector $\underline{\mathrm{v}}_k(s)$ can be written in the form:

$$\underline{\mathrm{v}}_k(s) = \mathbb{X}_k\underline{a}(s) \quad (30)$$

where the diagonal matrix $\mathbb{X}_{k+1}$ can be obtained recursively for $2 \leq k \leq N-1$ as:

$$\mathbb{X}_{k+1} = \frac{j}{\|\underline{\mathrm{v}}_k\|}\mathbb{X}_k\mathbb{R} + \frac{\|\underline{\mathrm{v}}_k\|}{\|\underline{\mathrm{v}}_{k-1}\|}\mathbb{X}_{k-1} - \frac{j}{\|\underline{\mathrm{v}}_k\|^3}\operatorname{Tr}\left\{\mathbb{X}_k^H\mathbb{X}_k\mathbb{R}\right\}\mathbb{X}_k \quad (31)$$

with initial conditions:

$$\mathbb{X}_1 = j\mathbb{R} \quad (32)$$

$$\mathbb{X}_2 = j\mathbb{X}_1\mathbb{R} - j\operatorname{Tr}\left\{\mathbb{X}_1^H\mathbb{X}_1\mathbb{R}\right\}\mathbb{X}_1 \quad (33)$$

Furthermore, it is shown that the entries of the complex Cartan matrix, $\mathbb{C}(s)$, for all $k = 1, 2, \ldots, N$ can be obtained as follows:

$$\|\underline{\mathrm{v}}_k\| = \sqrt{\mathrm{Tr}\left\{\mathbb{X}_k^H \mathbb{X}_k\right\}} \tag{34}$$

$$\underline{u}_k^H \underline{u}_k' = \frac{j}{\|\underline{\mathrm{v}}_k\|^2} \mathrm{Tr}\left\{\mathbb{X}_k^H \mathbb{X}_k \mathbb{R}\right\} \tag{35}$$

It is evident from Equations 34 - 35 that the complex Cartan matrix is constant (independent of s), confirming that the manifold is of hyperhelical shape.

C. Symmetric Linear Arrays

Before proceeding with our discussion of hyperhelical manifold curves, it is interesting to address the special case of linear arrays whose sensors exist in symmetric pairs about the origin. Specifically, it will be shown that, for this class of arrays, the diagonal entries of the complex Cartan matrix vanish and so the manifold admits a full representation in $\mathfrak{R}^N$. As a result, the Cartan matrix and moving frame will be seen to coincide exactly with those obtained under the previous framework.

To prove that the diagonal entries of $\mathbb{C}$ vanish (i.e. $\underline{u}_k^H \underline{u}_k' = 0$ for all $k = 1, 2, \ldots, N$), the following property of symmetric linear arrays can be used:

$$\mathrm{Tr}\left\{\mathbb{R}^i\right\} = 0, \quad \forall i \ odd \tag{36}$$

Therefore, in order to prove that $\underline{u}_k^H \underline{u}_k' = \frac{j}{\|\underline{\mathrm{v}}_k\|^2} \mathrm{Tr}\left\{\mathbb{X}_k^H \mathbb{X}_k \mathbb{R}\right\} = 0$, it is sufficient to prove that $\mathbb{X}_k$ is a polynomial in $\mathbb{R}$ of either strictly even or strictly odd powers.

Given that $\mathbb{X}_1 = j\mathbb{R}$, it naturally follows that:

$$\mathrm{Tr}\left\{\mathbb{X}_1^H \mathbb{X}_1 \mathbb{R}\right\} = 0 \tag{37}$$

From Equation 33, this leads to the simplified expression $\mathbb{X}_2 = j\mathbb{X}_1\mathbb{R} = -\mathbb{R}^2$ and so:

$$\mathrm{Tr}\left\{\mathbb{X}_2^H \mathbb{X}_2 \mathbb{R}\right\} = 0 \tag{38}$$

Using $\mathbb{X}_1 = j\mathbb{R}$ (a polynomial of strictly odd power) and $\mathbb{X}_2 = -\mathbb{R}^2$ (a polynomial of strictly even power) as initial conditions in Equation 31 will then lead to the desired result. Specifically, given $\mathbb{X}_k$, a polynomial of strictly odd powers of $\mathbb{R}$ and $\mathbb{X}_{k-1}$, a polynomial of strictly even powers of $\mathbb{R}$, Equation 31 becomes:

$$\mathbb{X}_{k+1} = \frac{j}{\|\underline{\mathrm{v}}_k\|} \underbrace{\mathbb{X}_k \mathbb{R}}_{\text{even order}} + \frac{\|\underline{\mathrm{v}}_k\|}{\|\underline{\mathrm{v}}_{k-1}\|} \underbrace{\mathbb{X}_{k-1}}_{\text{even order}} \tag{39}$$

which is a polynomial in $\mathbb{R}$ of strictly even powers. Obviously, equivalent reasoning can be applied to the converse case ($\mathbb{X}_k$ as a polynomial of strictly *even* powers and $\mathbb{X}_{k-1}$ a polynomial of strictly *odd* powers). Therefore, it follows in general that:

$$\mathrm{Tr}\left\{\mathbb{X}_k^H \mathbb{X}_k \mathbb{R}\right\} = 0 \tag{40}$$

which confirms that $\underline{u}_k^H \underline{u}_k' = 0$ for all $k = 1, 2, \ldots, N$. Using this result with the expressions derived in Appendix A reveals an exact equivalence with the previous framework in the special case of symmetric linear arrays (which exhibit zero "inclination angle"). In particular, a rearrangement of Equation 81 in terms of $\underline{u}_{k+1}(s)$ can be seen to coincide precisely with [5, Eq. 2.11]. A numerical example demonstrating this equivalence will be given in Section III-F.

D. Frame Matrix and Radii Vector

Consider a point $s \neq 0$ on the curve, $\underline{a}(s)$, whose local orthonormal basis, $\mathbb{U}(s)$, is unknown. Given the set of orthonormal vectors, $\mathbb{U}(s)$, at $s = 0$, namely $\mathbb{U}(0)$, a continuous differentiable transformation matrix $\mathbb{F}(s) \in \mathfrak{C}^{N \times N}$ must exist which relates $\mathbb{U}(s)$ to $\mathbb{U}(0)$ such that

$$\mathbb{U}(s) = \mathbb{U}(0)\, \mathbb{F}(s) \tag{41}$$

This transformation matrix $\mathbb{F}(s)$ is called the *frame matrix*, and is a nonsingular matrix which must satisfy the initial condition

$$\mathbb{F}(0) = \mathbb{I}_N \tag{42}$$

For hyperhelical array manifolds (for which the complex Cartan matrix is constant) the differential equation of Equation 17 is reduced to the following first order differential equation:

$$\mathbb{U}'(s) = \mathbb{U}(s)\mathbb{C} \tag{43}$$

which admits a solution of the type:

$$\mathbb{U}(s) = \mathbb{U}(0) \,\mathrm{expm}\left\{s\mathbb{C}\right\} \tag{44}$$

where $\mathbb{U}(0)$ accounts for initial conditions and $\mathrm{expm}\{\cdot\}$ denotes the matrix exponential. The frame matrix of Equation 41 is therefore given by:

$$\mathbb{F}(s) = \mathrm{expm}\left\{s\mathbb{C}\right\} \tag{45}$$

which clearly satisfies the condition $\mathbb{F}(0) = \mathbb{I}_N$. Since $\mathbb{F}(s)$ is, in general, complex-valued, its properties are subtly different from the analogous frame matrix derived under the previous framework [5, p.29]. Firstly, due to the skew-Hermitian nature of $\mathbb{C}$, it can be deduced that:

$$\mathbb{F}^H(s) = \mathbb{F}(-s) = \mathbb{F}^{-1}(s) \tag{46}$$

and so $\mathbb{F}(s)$ is unitary:

$$\mathbb{F}^H(s)\mathbb{F}(s) = \mathbb{I}_N \tag{47}$$

Using Equations 41 and 43, the complex Cartan matrix may be written as a function of the frame matrix:

$$\mathbb{C} = \mathbb{F}^H(s)\mathbb{F}'(s) \tag{48}$$

and so a similar first order differential equation to Equation 43 relates the frame and Cartan matrices:

$$\mathbb{F}'(s) = \mathbb{F}(s)\mathbb{C} \tag{49}$$

The manifold radii vector, $\underline{\mathcal{R}}(s) = [\mathcal{R}_1(s), \ldots, \mathcal{R}_N(s)]^T \in \mathfrak{C}^N$, is defined as:

$$\underline{\mathcal{R}}(s) = \mathbb{U}^H(s)\underline{a}(s) \tag{50}$$

such that its elements constitute the inner products of the manifold vector and the coordinate vectors

$$\mathcal{R}_k(s) = \underline{u}_k^H(s)\underline{a}(s) \tag{51}$$

for $k = 1, \ldots, N$. In the case of a hyperhelical manifold, $\mathcal{R}_k(s)$ can be seen to be independent of arc length:

$$\begin{aligned}\mathcal{R}_k &= \left(\frac{\mathbb{X}_k \underline{a}(s)}{\|\underline{v}_k\|}\right)^H \underline{a}(s) \\ &= \frac{\mathrm{Tr}\left\{\mathbb{X}_k^*\right\}}{\|\underline{v}_k\|}\end{aligned} \tag{52}$$

From Equation 50, the manifold vector may be written as a function of the strictly orthonormal basis and complex Cartan matrix:

$$\begin{aligned}\underline{a}(s) &= \mathbb{U}(s)\underline{\mathcal{R}}(s) \\ &= \mathbb{U}(0)\,\mathrm{expm}\{s\mathbb{C}\}\,\underline{\mathcal{R}}(s)\end{aligned} \tag{53}$$

Further examination of the recursive Equation 31 allows the following pattern to be observed:

$$\begin{array}{lr}\mathbb{X}_1 = j\mathbb{R} & imag. \\ \mathbb{X}_2 = j\mathbb{X}_1\mathbb{R} - j\,\mathrm{Tr}\left\{\mathbb{X}_1^H\mathbb{X}_1\mathbb{R}\right\}\mathbb{X}_1 & real \\ \mathbb{X}_3 = \frac{j}{\|\underline{v}_2\|}\mathbb{X}_2\mathbb{R} + \frac{\|\underline{v}_2\|}{\|\underline{v}_1\|}\mathbb{X}_1 - \frac{j}{\|\underline{v}_2\|^3}\,\mathrm{Tr}\left\{\mathbb{X}_2^H\mathbb{X}_2\mathbb{R}\right\}\mathbb{X}_2 & imag. \\ \mathbb{X}_4 = \ldots & real\end{array}$$

Indeed, it is straightforward to prove by induction that:

$$\mathbb{X}_k = \begin{cases} real, & \forall k \ even \\ imaginary, & \forall k \ odd\end{cases} \tag{54}$$

Using this result in conjunction with Equation 52 leads to:

$$\mathcal{R}_k = \begin{cases} real, & \forall k \ even \\ imaginary, & \forall k \ odd\end{cases} \tag{55}$$

In other words, consecutive entries of the radii vector toggle between purely real and purely imaginary.

Finally, the length of the radii vector is given by:

$$\begin{aligned}\|\underline{\mathcal{R}}(s)\| &= \sqrt{\underline{a}^H(s)\mathbb{U}(s)\mathbb{U}^H(s)\underline{a}(s)} \\ &= \sqrt{N}\end{aligned} \tag{56}$$

which corresponds to the radius of the hypersphere upon which the manifold curve lies, as might be expected.

E. Cartan Matrix Eigendecomposition

We begin by differentiating Equation 16a and substituting for $\underline{v}_k$ as follows:

$$\underline{u}_k'(s) = \frac{\mathbb{X}_k \underline{a}'(s)}{\|\underline{v}_k\|} \tag{57}$$

$$= j\mathbb{R}\frac{\mathbb{X}_k \underline{a}(s)}{\|\underline{v}_k\|} \tag{58}$$

$$= j\mathbb{R}\underline{u}_k(s) \tag{59}$$

Collecting all $\underline{u}_k'(s)$ for $k = 1, 2, \ldots, N$ therefore gives:

$$\mathbb{U}'(s) = j\mathbb{R}\mathbb{U}(s) \tag{60}$$

Combining Equations 43 and 60 leads to:

$$\begin{aligned}&j\mathbb{R}\mathbb{U}(s) = \mathbb{U}(s)\mathbb{C} \\ &\Rightarrow \mathbb{C} = j\mathbb{U}^H(s)\mathbb{R}\mathbb{U}(s)\end{aligned} \tag{61}$$

which reveals that the eigenvalues of $\mathbb{C}$ are equal to the normalised sensor positions of the (equivalent) linear array multiplied by the pure imaginary number, j, and the eigenvectors are given by $\mathbb{U}(s)$.

Note that, from Equation 61, it is possible to deduce that for any positive integer, $m > 0$:

$$\begin{aligned}\mathbb{C}^m &= \left[\mathbb{U}^H(s)\,(j\mathbb{R})\,\mathbb{U}(s)\right]^m \\ &= \left[\mathbb{U}^H(s)\,(j\mathbb{R})\,\underbrace{\mathbb{U}(s)\mathbb{U}^H(s)}_{=\mathbb{I}_N}\,(j\mathbb{R})\,\mathbb{U}(s)\ldots\right] \\ &= \mathbb{U}^H(s)\,(j\mathbb{R})^m\,\mathbb{U}(s)\end{aligned} \tag{62}$$

It is worth drawing attention to the fact that Equation 60 is a first order differential equation similar to Equation 43, except here $\mathbb{U}(s)$ is *pre*-multiplied by a constant matrix. The solution to this differential equation takes the form:

$$\mathbb{U}(s) = \mathrm{expm}\{js\mathbb{R}\}\,\mathbb{U}(0) \tag{63}$$

Therefore, a pre-multiplying frame matrix, $\mathbb{E}(s)$, is obtained (as an alternative to post-multiplying $\mathbb{F}(s)$) such that:

$$\mathbb{U}(s) = \mathbb{E}(s)\mathbb{U}(0) \tag{64}$$

with $\mathbb{E}(s) \triangleq \mathrm{expm}\{js\mathbb{R}\}$ which has initial condition $\mathbb{E}(0) = \mathbb{I}_N$. In fact, it can be seen that $\mathbb{E}(s)$ is also the matrix which relates the original manifold vector $\underline{a}(0)$ to any other manifold vector:

$$\begin{aligned}\underline{a}(s) &= \mathbb{U}(s)\underline{\mathcal{R}}(s) \\ &= \mathbb{E}(s)\mathbb{U}(0)\underline{\mathcal{R}}(s) \\ &= \mathbb{E}(s)\underline{a}(0)\end{aligned} \tag{65}$$

F. Numerical Examples

For the first example, consider a (non-symmetric) linear array of four sensors, i.e. $N = 4$, with sensor positions in half-wavelengths given by $\underline{r}_x = [-2.1, -1.1, 0.9, 2.3]^T$. Assuming $\phi_0 = 0$, the transformation matrices, $\mathbb{X}_n$, for $n = 1, \ldots, 4$, and the Cartan matrix, $\mathbb{C}$, are given by:

$$\mathbb{X}_1 = \begin{bmatrix} -j0.6134, & 0, & 0, & 0 \\ 0, & -j0.3213, & 0, & 0 \\ 0, & 0, & +j0.2629, & 0 \\ 0, & 0, & 0, & +j0.6718 \end{bmatrix}$$

$$\mathbb{X}_2 = \begin{bmatrix} -0.4115, & 0, & 0, & 0 \\ 0, & -0.1217, & 0, & 0 \\ 0, & 0, & -0.0540, & 0 \\ 0, & 0, & 0, & -0.4128 \end{bmatrix}$$

$$\mathbb{X}_3 = \begin{bmatrix} +j0.0682, & 0, & 0, & 0 \\ 0, & -j0.1229, & 0, & 0 \\ 0, & 0, & +j0.1351, & 0 \\ 0, & 0, & 0, & -j0.0494 \end{bmatrix}$$

$$\mathbb{X}_4 = \begin{bmatrix} 0.0589, & 0, & 0, & 0 \\ 0, & -0.2181, & 0, & 0 \\ 0, & 0, & -0.2158, & 0 \\ 0, & 0, & 0, & 0.0339 \end{bmatrix}$$

$$\mathbb{C}_{\mathrm{r}} = \begin{bmatrix} 0, & -0.6006, & 0, & 0, & 0, & 0, & 0, & 0 \\ +0.6006, & 0, & -0.2140, & 0, & 0, & 0, & 0, & 0 \\ 0, & +0.2140, & 0, & -0.3534, & 0, & 0, & 0, & 0 \\ 0, & 0, & +0.3534, & 0, & -0.2357, & 0, & 0, & 0 \\ 0, & 0, & 0, & +0.2357, & 0, & -0.5588, & 0, & 0 \\ 0, & 0, & 0, & 0, & +0.5588, & 0, & -0.1209, & 0 \\ 0, & 0, & 0, & 0, & 0, & +0.1209, & 0, & -0.2935 \\ 0, & 0, & 0, & 0, & 0, & 0, & +0.2935, & 0 \end{bmatrix} \quad (67)$$

and

$$\mathbb{C} = \begin{bmatrix} +j0.0574, & -0.5979, & 0, & 0 \\ +0.5979, & +j0.0185, & -0.2011, & 0 \\ 0, & +0.2011, & -j0.0314, & -0.3143 \\ 0, & 0, & +0.3143, & -j0.0445 \end{bmatrix} \quad (66)$$

Using the traditional framework, the purely real Cartan matrix of the same array is the 8×8 real matrix shown in Equation 67, above.

Comparing the results obtained for the two Cartan matrices, it is clear that the first curvature of $\mathbb{C}_{\mathrm{r}}$ (namely $\kappa_1 = 0.6006$) can be obtained as the magnitude of the first two elements of $\mathbb{C}$, i.e. $\kappa_1 = \left|\|\underline{v}_2\| + \underline{u}_1^H \underline{u}_1'\right| = |0.5979 + j0.0574|$. Subsequent real curvatures $\kappa_2, \kappa_3, \ldots, \kappa_{d-1}$ may be obtained via more complicated expressions involving the lower order entries of the complex Cartan matrix (but not vice versa, since $\mathbb{C}$ contains information regarding inclination angle, which is absent from $\mathbb{C}_{\mathrm{r}}$).

For the second example, consider the *symmetric* linear array $\underline{r}_x = [-2.3, -1.1, 1.1, 2.3]^T$. Assuming $\phi_0 = 0$, the complex Cartan matrix of its manifold is given by:

$$\mathbb{C} = \begin{bmatrix} 0, & -0.5903, & 0, & 0 \\ +0.5903, & 0, & -0.2069, & 0 \\ 0, & +0.2069, & 0, & -0.3297 \\ 0, & 0, & +0.3297, & 0 \end{bmatrix} \quad (68)$$

and the radii vector $\underline{\mathcal{R}} = [0, -1.6939, 0, -1.0633]^T$. These results, and indeed the orthonormal basis $\mathbb{U}(s)$, computed at any azimuth, θ_0, are found to coincide exactly with those obtained through the previous curvilinear differential geometry framework for the same symmetric array:

$$\mathbb{C}_{\mathrm{r}} = \mathbb{C} \quad (69)$$

$$\mathbb{U}_{\mathrm{w}}(s) = \mathbb{U}(s) \quad (70)$$

The convergence of the two frameworks for symmetric linear arrays, where the manifold dimensionality is reduced to real only, demonstrates that the two approaches are fundamentally correct and that the difference lies in the handling of the purely imaginary dimensions.

IV. Conclusion

In this paper, a new theoretical framework has been presented which provides an alternative to the existing differential geometric approach to analysing array manifold curves. While the traditional approach restricted the entries of the Cartan matrix to be purely real, the framework proposed in this paper incorporated a complex, skew-Hermitian Cartan matrix. In this way, a strictly orthogonal moving frame has been established, which facilitates a more convenient and direct method of analysis.

A number of specific features of the proposed framework have been highlighted and compared to the traditional approach (with particular attention given to hyperhelical array manifold curves). Concepts such as the radius of circular approximation, manifold curve radii vector and the frame matrix were revisited and rederived under the new framework.

Appendix A
Derivation of the Complex Cartan Matrix

In order to evaluate the elements of the complex Cartan matrix, Equation 16b is first rearranged, following a change of variables, as:

$$\underline{u}_k'(s) = \underline{\mathrm{v}}_{k+1}(s) + \sum_{m=1}^{k} \left(\underline{u}_m^H(s)\underline{u}_k'(s)\right) \underline{u}_m(s) \quad (71)$$

Given that $\underline{u}_m^H(s)\underline{u}_k(s) = 0$ for $m = 1, \ldots, k-1$ with $2 \leq k \leq N-1$, it follows that:

$$\frac{d}{ds}\left[\underline{u}_m^H(s)\underline{u}_k(s)\right] = 0$$
$$\Rightarrow \underline{u}_m^H(s)\underline{u}_k'(s) = -\underline{u}_m'^H(s)\underline{u}_k(s) \quad (72)$$

Substituting this result into Equation 71 yields for $2 \leq k \leq N-1$:

$$\underline{u}_k'(s) = \underline{\mathrm{v}}_{k+1}(s) - \sum_{m=1}^{k} \left(\underline{u}_m'^H(s)\underline{u}_k(s)\right) \underline{u}_m(s) \quad (73)$$

or, by a simple change of variables:

$$\underline{u}_m'(s) = \underline{\mathrm{v}}_{m+1}(s) - \sum_{i=1}^{m} \left(\underline{u}_i'^H(s)\underline{u}_m(s)\right) \underline{u}_i(s) \quad (74)$$

Substituting this expression for $\underline{u}_m'(s)$ back into Equation 73 then leads to (note that (s) is dropped for simplicity for the next few equations):

$$\underline{u}_k' = \underline{\mathrm{v}}_{k+1} - \sum_{m=1}^{k} \left[\underline{\mathrm{v}}_{m+1} - \sum_{i=1}^{m} \underline{u}_i'^H \underline{u}_m \underline{u}_i\right]^H \underline{u}_k \underline{u}_m \quad (75)$$

This equation can be rewritten as:

$$\underline{u}'_k = \underline{\mathrm{v}}_{k+1} - \sum_{l=1}^{k} \left(\underline{\mathrm{v}}_{l+1}^H \underline{u}_k\right) \underline{u}_l + \sum_{m=1}^{k} \sum_{i=1}^{m} \left(\underline{u}_m^H \underline{u}'_i\right) \left(\underline{u}_i^H \underline{u}_k\right) \underline{u}_m \tag{76}$$

Then, noting that:

$$\underline{\mathrm{v}}_{l+1}^H \underline{u}_k \neq 0 \quad \text{iff } l = k-1 \tag{77}$$

$$\underline{u}_i^H \underline{u}_k \neq 0 \quad \text{iff } i = k \tag{78}$$

allows Equation 76 to be simplified to:

$$\underline{u}'_k = \underline{\mathrm{v}}_{k+1} - \|\underline{\mathrm{v}}_k\| \, \underline{u}_{k-1} + \left(\underline{u}_k^H \underline{u}'_k\right) \underline{u}_k \tag{79}$$

Finally, noting that $\underline{\mathrm{v}}_k = \|\underline{\mathrm{v}}_k\| \, \underline{u}_k$, the method for calculating $\underline{u}'_k(s)$ can now be stated in full.

For $k = 1$:

$$\underline{u}'_1 = \left(\underline{u}_1^H \underline{u}'_1\right) \underline{u}_1 + \|\underline{\mathrm{v}}_2\| \, \underline{u}_2 \tag{80}$$

For $2 \leq k \leq N-1$:

$$\underline{u}'_k = -\|\underline{\mathrm{v}}_k\| \, \underline{u}_{k-1} + \left(\underline{u}_k^H \underline{u}'_k\right) \underline{u}_k + \left\|\underline{\mathrm{v}}_{k+1}\right\| \underline{u}_{k+1} \tag{81}$$

For $k = N$:

$$\underline{u}'_N = -\|\underline{\mathrm{v}}_N\| \, \underline{u}_{N-1} + \left(\underline{u}_N^H \underline{u}'_N\right) \underline{u}_N \tag{82}$$

where Equation 80 is obtained by direct rearrangement of Equation 73 (with $k = 1$). Meanwhile, Equation 82 follows from the fact that attempting to evaluate $\underline{\mathrm{v}}_{N+1}$ (in Equation 16b) must yield the zero vector.

Using Equations 80 - 82 with Equation 17 leads directly to $\mathbb{C}(s)$ in Equation 18.

Appendix B
Properties of Hyperhelical Array Manifolds

In order to analyse the differential geometry of the hyperhelical array manifold curve, the parameters presented in Section II are next evaluated for the special array manifold vector structure of Equation 24. Specifically, in order to identify the orthonormal basis vectors, $\underline{u}_k(s)$, and elements of the Cartan matrix, expressions will be developed for $\underline{\mathrm{v}}_k(s)$ and $\underline{u}_k^H \underline{u}'_k$.

It will be shown in this appendix that, in the case of hyperhelical array manifolds, $\underline{\mathrm{v}}_k(s)$ can be written in the form:

$$\underline{\mathrm{v}}_k(s) = \mathbb{X}_k \underline{a}(s) \tag{83}$$

To begin, we write the first (unnormalised) frame vector as:

$$\underline{\mathrm{v}}_1(s) = \underbrace{j\mathbb{R}}_{\mathbb{X}_1} \, \underline{a}(s) \tag{84}$$

Clearly, $\|\underline{\mathrm{v}}_1(s)\|$ is independent of arc length, s:

$$\begin{aligned}\|\underline{\mathrm{v}}_1\| &= \sqrt{\mathrm{Tr}\left\{\mathbb{X}_1^H \mathbb{X}_1\right\}} = \sqrt{\mathrm{Tr}\left\{\mathbb{R}^T \mathbb{R}\right\}} \\ &= 1\end{aligned} \tag{85}$$

and so the first frame vector derivative may be obtained as:

$$\begin{aligned}\underline{u}'_1(s) &= \frac{d}{ds} \frac{\mathbb{X}_1 \underline{a}(s)}{\|\underline{\mathrm{v}}_1\|} = \frac{\mathbb{X}_1}{\|\underline{\mathrm{v}}_1\|} \underline{a}'(s) \\ &= \frac{j}{\|\underline{\mathrm{v}}_1\|} \mathbb{X}_1 \mathbb{R} \underline{a}(s)\end{aligned} \tag{86}$$

(where $\|\underline{\mathrm{v}}_1\| = 1$ has not been deleted for the sake of consistency as k is increased). Next, it can be seen that the first element of the complex Cartan matrix, $\underline{u}_1^H(s) \underline{u}'_1(s)$, is also independent of s:

$$\begin{aligned}\underline{u}_1^H \underline{u}'_1 &= \frac{j}{\|\underline{\mathrm{v}}_1\|^2} \underline{a}^H(s) \mathbb{X}_1^H \mathbb{X}_1 \mathbb{R} \underline{a}(s) \\ &= \frac{j}{\|\underline{\mathrm{v}}_1\|^2} \mathrm{Tr}\left\{\mathbb{X}_1^H \mathbb{X}_1 \mathbb{R}\right\}\end{aligned} \tag{87}$$

Proceeding to $k = 2$ and substituting from the above expressions leads to:

$$\begin{aligned}\underline{\mathrm{v}}_2(s) &= \underline{u}'_1(s) - \left(\underline{u}_1^H \underline{u}'_1\right) \underline{u}'_1(s) \\ &= \underbrace{\left[\frac{j}{\|\underline{\mathrm{v}}_1\|} \mathbb{X}_1 \mathbb{R} - \frac{j}{\|\underline{\mathrm{v}}_1\|^3} \mathrm{Tr}\left\{\mathbb{X}_1^H \mathbb{X}_1 \mathbb{R}\right\} \mathbb{X}_1\right]}_{\mathbb{X}_2} \underline{a}(s)\end{aligned} \tag{88}$$

and continuing as before allows the second diagonal entry of the Cartan matrix to be obtained:

$$\underline{u}_2^H \underline{u}'_2 = \frac{j}{\|\underline{\mathrm{v}}_2\|^2} \mathrm{Tr}\left\{\mathbb{X}_2^H \mathbb{X}_2 \mathbb{R}\right\} \tag{89}$$

Proceeding in this manner leads the pattern stated in Equations 31 and 35 to emerge. Proof by induction is a straightforward extension of the previous discussion.

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